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Solvable Eight-Vertex Model on an Arbitrary Planar Lattice

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SOLVABLE EIGHT-VERTEX MODEL ON AN ARBITRARY PLANAR LATTICE

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CONTENTS

Any planar set of intersecting straight lines forms a four-coordinated graph, or 'lattice', provided no three lines intersect at a point. For any such lattice an eight-vertex model can be constructed. Provided the interactions satisfy certain constraints (which are in general temperature-dependent), the model can be solved exactly in the thermodynamic limit, its local properties at a particular site being those of a related square lattice.

A particular case is a solvable model on the Kagomé lattice. It is shown that this model includes as special cases many of the models in statistical mechanics that have been solved exactly, notably the square, triangular and honeycomb Ising models, and the square eight-vertex model.

Some remarkable equivalences between correlations on different lattices are also established.

1. Introduction

There are a number of two-dimensional statistical mechanical models of interacting systems for which the free energy has been evaluated exactly in the thermodynamic limit. In particular, the following models have been solved in the absence of magnetic or electric fields:

(i) the translation-invariant Ising model on the square lattice (Onsager 1944);

(ii) the translation-invariant Ising model on the triangular or honeycomb lattice (Houtappel 1950; Husimi & Syozi 1950; Wannier 1950; Stephenson 1964);

(iii) the ice-type ferroelectric models on the square lattice (Lieb 1967; Sutherland 1967);

(iv) the eight-vertex model on the square lattice (Baxter 1972);

(v) the three-spin model on the triangular lattice (Wood & Griffiths 1972; Baxter & Wu 1974);

(vi) an ice-type model on the triangular lattice satisfying certain temperature-dependent restrictions (Baxter 1969; Kelland 1974); this has recently been shown to be equivalent to a restricted ice-type model on the Kagomé lattice (Baxter, Temperley & Ashley 1977). Both (iii) and (vi) are also equivalent to Potts models at their transition temperatures, on the square and triangular lattices, respectively (see also Baxter, Kelland & Wu 1976).

From a mathematical point of view, all these models have two common features. One is that their solution leads sooner or later to the introduction of elliptic functions. (In the ice-type models these functions occur in the distribution of the wavenumbers $k_1, \ldots, k_n$ (Baxter 1971, appendix)). The other common feature is that they can all be solved by some appropriately generalized Bethe ansatz (see for example, Baxter 1973).

This suggests that there may be some more general model which includes all of (i) to (vi) as special cases. Considerable progress has already been made in this direction, for by construction (iv) contains (i) and (iii) as special cases, and has also been shown to include (v) (Baxter & Enting 1976). However, it does not include (ii) or (vi).

In this paper a general model including (i)–(vi) is presented. If some reasonable assumptions are made concerning the thermodynamic limit, then the properties of the model can be obtained directly from the known or conjectured results for (iv).

It should be noted that there are very few exactly solved two-dimensional models that are not included in the list (i) to (vi). The only ones that come to mind are the spherical model, non-translation-invariant Ising models, ice-type models in the presence of direct electric fields, and colouring problems on the honeycomb and square lattices (Baxter 1970*a*, *b*). There are basic mathematical differences between the solutions of the first three and those of models (i) to (vi), so in this sense they are 'exceptional' models.

The two solved colouring problems share the same common mathematical features as (i) to (vi) and should probably be listed with them. However, they are defined very differently and it is not yet obvious that they are equivalent to any special case of the general model discussed in this paper.

2. The general model

A model that includes (i) to (vi) can be constructed on the Kagomé lattice, as is shown in §8. However, it is actually simpler and more illuminating to consider a yet more general model, namely a restricted eight-vertex model on an arbitrary lattice of intersecting straight lines.

Consider some simply connected convex planar region, such as the interior of a circle, and draw N straight lines within it, starting and ending at the boundary. No three lines are allowed to intersect at a common point.

Two typical sets of such lines are shown in figure 1. The intersections of these lines from the sites of a graph, or 'lattice' $\mathscr{L}$. The line segments between sites form the edges of $\mathscr{L}$. Each site is the endpoint of four edges.

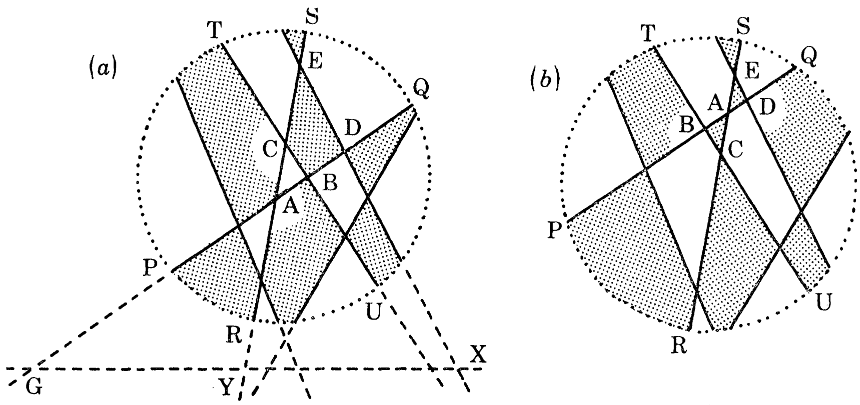

Figure 1. Typical irregular straight-line graphs, or 'lattices' $\mathscr{L}$. The second differs from the first only in that the line PQ has been shifted upwards. The broken lines in the first figure are a possible base line, with the lattice lines extended to cross it.

Consider two intersecting lines, such as PQ and RS in figure 1. Let A be their point of intersection. To the site A assign an interaction coefficient K''_{A}, to the angle PAR a coefficient K_{PAR}, and to the angle RAQ another coefficient K_{RAQ}. Make no distinction between opposite angles at an intersection, or between the senses of an angle, so that for example K_{PAR}, K_{RAP}, K_{QAS}, K_{SAQ} are all identical.

Do this for every intersecting pair of lines. Thus if all lines intersect there will be $\frac{1}{2}N(N-1)$ interaction coefficients associated with sites, and $N(N-1)$ coefficients associated with angles.

We now construct an eight-vertex model on $\mathscr{L}$, with K''_{A}, K_{PAR}, etc. as interaction coefficients. To do this we first label the *faces* of $\mathscr{L}$ in some way, and with each face l we associate a spin σ_l. Each such spin has values $+1$ or -1.

Define an Hamiltonian $\mathscr{H}$ by

$$-\beta\mathscr{H} = \Sigma[K_{PAR}\,\sigma_l\,\sigma_n + K_{RAQ}\,\sigma_m\,\sigma_p + K''_{A}\,\sigma_l\,\sigma_m\,\sigma_n\,\sigma_p], \tag{2.1}$$

where the summation is over all sites A of $\mathscr{L}$, and for each term in the summation l, m, n, p are the four faces surrounding the site, arranged as in figure 2 so that l and n are opposite, the angle PAR is a corner of either face l or face n, RAQ is a corner of either face m or face p.

The object of statistical mechanics is to calculate the partition function

$$Z = \sum_{\sigma} e^{-\beta \mathscr{H}} \tag{2.2}$$

(the σ-summation being over all values of all spins in the lattice) and various thermodynamic averages, such as the two-spin correlation

$$\langle \sigma_l \sigma_m \rangle = Z^{-1} \sum_{\sigma} \sigma_l \sigma_m e^{-\beta \mathscr{H}}. \tag{2.3}$$

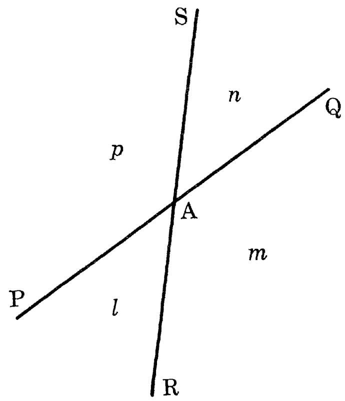

FIGURE 2. A typical site A of $\mathscr{L}$, showing the surrounding faces l, m, n, p, ordered as in equation (2.1).

As is indicated in figure 1 the faces of $\mathscr{L}$ can be grouped into two classes X and Y (shaded and unshaded) so that no two faces of the same class have an edge in common. (An equivalent statement is that the dual lattice of $\mathscr{L}$ is bipartite.) The two-spin interactions in (2.1) link only faces of the same class, so (2.1) is the sum of three terms:

(*a*) a nearest-neighbour two-spin Ising Hamiltonian defined on the class X faces,

(*b*) a similar Hamiltonian defined on the class Y faces,

(*c*) a purely four-spin Hamiltonian coupling the X and Y spins, with coefficients K''_{A}.

In particular, if the K''_{A} are all zero, then the model reduces to two independent ordinary Ising models, one on the X spins, the other on the Y spins.

It follows that (2.1) is a fairly obvious generalization to a rather arbitrary planar lattice of the eight-vertex model, which was originally defined on the square lattice (Fan & Wu 1970).

It would be marvellous to calculate Z for any choice of the lattice coefficients K''_{A}, K_{PAR}, etc., but the author knows of no way to do this. What can be done is to calculate Z (in the thermodynamic limit when N becomes infinite) if the parameters satisfy the conditions given in §4.

3. Formulation as an eight-vertex model

In this paper the above Ising-type formulation will mostly be used. However, to connect with previous results it is sometimes desirable to regard (2.2) as the partition function of a ferroelectric-type model.

This can be done by using an argument due to Wu (1971) and Kadanoff & Wegner (1971). Every edge of $\mathscr{L}$ lies between a shaded (class X) and an unshaded (class Y) face. If the spins on the two faces are alike, draw an arrow on the edge so that an observer following the arrow has the shaded face on his left. If the spins are different, draw the arrow so that the observer has the shaded face on his right.

Now consider a site j of the lattice $\mathscr{L}$. There are 16 possible choices of the four surrounding spins. To each choice there corresponds a configuration of arrows on the four edges. Each arrow configuration corresponds to two spin configurations, one being obtained from the other by reversing all spins. Thus there are eight arrow configurations, as shown in figure 3. In each arrow configuration there are an even number of arrows pointing into the site (or vertex). We call this the *eight-vertex condition*. An arrow covering of the edges of $\mathscr{L}$ is 'allowed' if the eight-vertex condition is satisfied at every site.

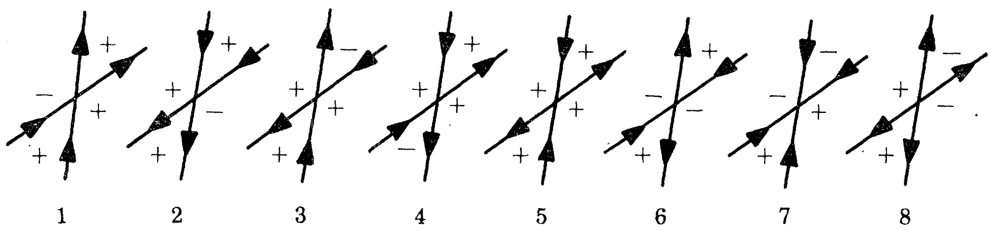

FIGURE 3. The eight arrow configurations allowed at a site. Corresponding spin configurations are also shown. The other eight spin configurations can be obtained by reversing all four spins.

To every configuration of spins on the faces of $\mathscr{L}$ there corresponds an allowed arrow covering of the edges. To every allowed arrow covering there correspond two spin configurations, one being obtained from the other by reversing all spins. Since the Hamiltonian (2.1) is unchanged by reversing all spins, it follows from (2.2) that for a lattice of M sites

$$Z = 2 \sum_{C} \omega_1 \omega_2 \ldots \omega_M, \tag{3.1}$$

where the sum is over all allowed arrow configurations C and ω_j is the Boltzmann weight of site j for configuration C.

Consider the site A, or j, shown in figure 2, and for brevity set

$$K_j = K_{\mathrm{RAQ}}, \quad K_j' = K_{\mathrm{PAR}}, \quad K_j' = K_{\mathrm{A}}''. \tag{3.2}$$

Then the Boltzmann weight ω_j is given by

$$\omega_j = \exp\,(K_j \sigma_m \sigma_p + K_j' \sigma_l \sigma_n + K_j'' \sigma_l \sigma_m \sigma_n \sigma_p). \tag{3.3}$$

From figure 3 it follows that

$\omega_j = a_j(b_j, c_j, d_j)$ if the arrows at site j are in configuration 1 or 2 (3 or 4, 5 or 6, 7 or 8), (3.4)

where

$$a_j, b_j, c_j, d_j = \exp\,(K_j - K_j' - K_j''), \quad \exp\,(-K_j + K_j' - K_j''),$$
$$\exp\,(K_j + K_j' + K_j''), \quad \exp\,(-K_j - K_j' + K_j''). \tag{3.5}$$

In figure 3 the top-right and bottom-left faces have been regarded as shaded, but reversing the shading merely reverses the arrows, which leaves the weights unchanged. Thus ω_j is always given by the above rule (3.4).

Duality

From (3.5), the site weights satisfy the normalization condition $a_j b_j c_j d_j = 1$. For some purposes it is convenient to ignore this requirement and regard a_j, b_j, c_j, d_j as independent variables.

(This is equivalent to re-defining the energy zero of the Hamiltonian (2.1).) Then Z is given by (3.1) and (3.4) and is a linear function of $a_1, a_2, \ldots, d_M$. Using the same weak-graph symmetry argument as that employed by Fan & Wu (1970), and Wegner (1973), for the square lattice, one can establish for the general lattice $\mathscr{L}$ that

$$Z(\{a_j^*, b_j^*, c_j^*, d_j^*\}) = Z(\{a_j, b_j, c_j, d_j\}), \tag{3.6}$$

where

$$\left.\begin{aligned} a_j^* &= \tfrac{1}{2}(a_j - b_j + c_j - d_j), \\ b_j^* &= \tfrac{1}{2}(-a_j + b_j + c_j - d_j), \\ c_j^* &= \tfrac{1}{2}(a_j + b_j + c_j + d_j), \\ d_j^* &= \tfrac{1}{2}(-a_j - b_j + c_j + d_j). \end{aligned}\right\} \tag{3.7}$$

This is a duality relation, taking a low-temperature system to a high-temperature one.

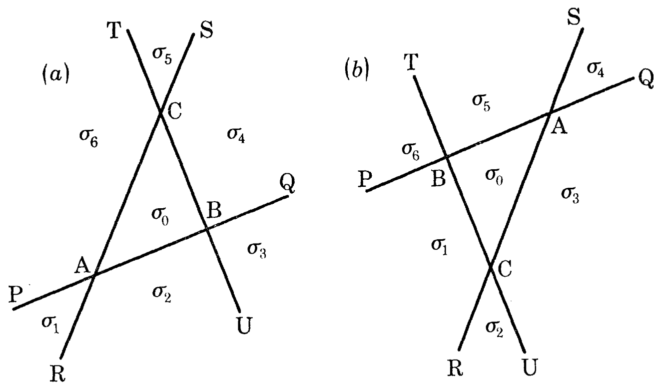

FIGURE 4. The triangles ABC in figures 1*a* and *b*, showing the positions of the interior spin σ_0 and the surrounding spins $\sigma_1, \ldots, \sigma_6$.

4. STAR-TRIANGLE CONDITIONS

Consider the two lattices shown in figure 1. They differ only in that the line PQ has been shifted from one side of site C to the other. This changes the triangle ABC, but leaves the rest of the lattice unaltered. In both lattices A is the intersection of the lines PQ and RS. Similarly B is the intersection of PQ and TU, C is the intersection of RS and TU.

Construct eight-vertex models as above on both lattices, using the same boundary conditions (e.g. all boundary spins up) and the same coefficients K''_{A}, K_{PAR}, etc.

For each model, let $\sigma_1, \ldots, \sigma_6$ be the six spins on the faces surrounding the triangle ABC, and σ_0 the spin inside the triangle, as indicated in figure 4. For brevity set

$$\left.\begin{aligned} K_1 &= K_{\mathrm{SAQ}}, & K_2 &= K_{\mathrm{PBT}}, & K_3 &= K_{\mathrm{UCR}}, \\ K'_1 &= K_{\mathrm{RAQ}}, & K'_2 &= K_{\mathrm{QBT}}, & K'_3 &= K_{\mathrm{TCR}}, \\ K''_1 &= K''_{\mathrm{A}}, & K''_2 &= K''_{\mathrm{B}}, & K''_3 &= K''_{\mathrm{C}}. \end{aligned}\right\} \tag{4.1}$$

Thus K_1, K_2, K_3 are the coefficients of the *interior* angles of the triangle ABC, while K'_1, K'_2, K'_3 are the coefficients of the *exterior* angles.

The summand in (2.2) is the same for both models, except the factors coming from the interactions round sites A, B, C. The centre spin σ_0 occurs in only these factors, so the summation over

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

σ_0 may in each case be performed to give a total Boltzmann weight for the triangle ABC. Doing this, the weight for the first model is

$$W_1 = 2\exp(K_1'\sigma_6\sigma_2 + K_2'\sigma_2\sigma_4 + K_3'\sigma_4\sigma_6) \times \cosh(K_1\sigma_1 + K_2\sigma_3 + K_3\sigma_5 + K_1''\sigma_1\sigma_6\sigma_2 + K_2''\sigma_3\sigma_2\sigma_4 + K_3''\sigma_5\sigma_4\sigma_6). \quad (4.2a)$$

while the weight for the second model is

$$W_2 = 2\exp(K_1'\sigma_3\sigma_5 + K_2'\sigma_5\sigma_1 + K_3'\sigma_1\sigma_3) \times \cosh(K_1\sigma_4 + K_2\sigma_6 + K_3\sigma_2 + K_1''\sigma_4\sigma_3\sigma_5 + K_2''\sigma_6\sigma_5\sigma_1 + K_3''\sigma_2\sigma_1\sigma_3). \quad (4.2b)$$

Both weights are functions of $\sigma_1, \ldots, \sigma_6$. The other factors in the summand of (2.2) are identical, for all values of $\sigma_1, \ldots, \sigma_6$, so the two partition functions will be the same if

$$W_1 = W_2 \quad (4.3)$$

for all $\sigma_1, \ldots, \sigma_6$. Further, averages such as (2.3) will also be the same, provided neither spin σ_l nor spin σ_m lies inside the triangle ABC.

Since $\sigma_1, \ldots, \sigma_6$ each have two values, (4.3) represents 64 equations. However, (4.3) is unchanged by negating all of $\sigma_1, \sigma_3, \sigma_5$, or all of $\sigma_2, \sigma_4, \sigma_6$, so the 64 equations reduce to 16. Further, (4.3) is unaltered by interchanging σ_1 with σ_4, σ_3 with σ_6, and σ_5 with σ_2. This means that there are only six distinct equations, namely

$$\exp(2K_j' + 2K_k') = \frac{\cosh(K_1 + K_2 + K_3 + K_i'' - K_j'' - K_k'')}{\cosh(-K_i + K_j + K_k - K_i'' + K_j'' + K_k'')} \quad (4.4)$$

$$\exp(2K_j' - 2K_k') = \frac{\cosh(K_i - K_j + K_k - K_i'' + K_j'' + K_k'')}{\cosh(K_i + K_j - K_k - K_i'' + K_j'' + K_k'')} \quad (4.5)$$

where (i, j, k) is any permutation of (1, 2, 3). There are three distinct equations of the form (4.4), three of form (4.5).

(It is tempting to try to generalize the present work by allowing $K_1, \ldots, K_3''$ to be different for the two lattices, but the interchange symmetry of (4.3) is then destroyed, leaving 16 apparently distinct equations. Thus one gains nine more degrees of freedom, but ten more apparent restrictions, and it appears that no such generalization is possible.)

The equations (4.4) and (4.5) have been reported before: they are equivalent to the commutation conditions for the square lattice eight-vertex model (Baxter 1972, eqn B8), and a special case has been discussed for the six-vertex models (Baxter, Temperley & Ashley 1977, eqn 111).

The six equations (4.4) and (4.5) are not independent. The K_1', K_2', K_3' can all be eliminated by taking ratios and products, giving three apparently distinct equations which can be regarded as defining K_1'', K_2'', K_3'' as functions of K_1, K_2, K_3. However, these three equations are identically satisfied if $K_1'' = K_2'' = K_3''$, so cannot be independent.

It appears that there is in general no other solution of these three equations, so a corollary of (4.4) and (4.5) is, taking K'' to be the common value,

$$K_1'' = K_2'' = K_3'' = K''. \quad (4.6)$$

If $K'' = 0$, then the equations (4.4) become the star-triangle relation between an Ising model on the honeycomb lattice, with interactions K_1, K_2, K_3, and an equivalent Ising model on the triangular lattice with interactions K_1', K_2', K_3' (Houtappel 1950, eqn 23). This follows directly from their derivation, since if $K'' = 0$ the weights W_1 and W_2 each factor into a function of $\sigma_1, \sigma_3, \sigma_5$ and a function of $\sigma_2, \sigma_4, \sigma_6$, and (4.3) factors into two independent star-triangle transformations with identical coefficients. Thus for $K'' \neq 0$ (4.3) can be regarded as a generalized star-triangle transformation.

By using (4.6), the equations (4.5) can be obtained from (4.4) by taking ratios. Thus (4.4) and (4.6) ensure that (4.3) is satisfied, i.e. Z is unchanged by shifting the line PQ across the site C. Remembering that $K_1, \ldots, K_3''$ are defined by (4.1) and figure 4, we call equations (4.4) and (4.6) the *star-triangle relations for the triangle* ABC. Remember that in these equations K_1, K_2, K_3 are the interaction coefficients assigned to the inside angles of the triangle ABC, while K_1', K_2', K_3' are assigned to the corresponding supplementary exterior angles.

Corollaries

Various corollaries of (4.4) and (4.6) can be obtained by eliminating two of the angle coefficients $K_1, \ldots, K_3'$. In particular, eliminating K_i between equations (4.4) and (4.5) as written, using (4.6), gives

$$\Delta_j = \Delta_k, \tag{4.7}$$

where

$$\Delta_j = -\sinh 2K_j \sinh 2K_j' - \tanh 2K'' \cosh 2K_j \cosh 2K_j'. \tag{4.8}$$

Since Δ_j is a symmetric function of the angle coefficients K_j, K_j' of a single site, it can be thought of as a 'site parameter' similar to K_j''. Since Δ_1 corresponds to the site A in figure 4, it can alternatively be written as Δ_A. Similarly, Δ_2 and Δ_3 can be written as Δ_B and Δ_C, respectively. From (4.7), they have a common value Δ. By using (4.1), it follows that (4.6) and (4.7) can be written as

$$K_A'' = K_B'' = K_C'' = K'', \quad \Delta_A = \Delta_B = \Delta_C = \Delta. \tag{4.9}$$

In the limit when $K_j, K_j', -K''$ all tend to plus infinity, their differences remaining constant, the eight-vertex model reduces to an 'ice-type' six-vertex model. The Δ defined above is then the same as that used by Lieb (1967).

Also, eliminating K_2' and K_3' gives the equations

$$-\cosh 2K_2 \cosh 2K_3 + \coth 2K_1' \sinh 2K_2 \sinh 2K_3 = \cosh 2K_1 \cosh 2K'' + \coth 2K_1' \sinh 2K_1 \sinh 2K''. \tag{4.10}$$

Two similar equations can be obtained by permuting the suffixes 1, 2, 3.

Quadrilateral theorem

Consider the quadrilateral shown in figure 5. Let K_A'', K_{DFB}, etc. be the interaction coefficients assigned to the six sites and twelve angles. Suppose that these satisfy the star-triangle relations (4.4) and (4.6) for each of the triangles AEF, BFD, CDE. Then it follows that they also satisfy the star-triangle relations for the triangle ABC.

A brute-force proof of this theorem is given in appendix A, by using only elementary algebra. A much neater proof is given in §6, but this makes use of elliptic functions. From the form of the equations (4.8) and (4.10), it seems likely that a third proof, of intermediate difficulty, could be obtained by using hyperbolic trigonometry (Onsager 1944, p. 135; Coxeter 1947, p. 238).

Note in particular that if the star-triangle relations are satisfied for AEF, BFD and CDE, then

$$\begin{gathered} K_A'' = K_B'' = K_C'' = K_D'' = K_E'' = K_F'' \\ \Delta_A = \Delta_B = \Delta_C = \Delta_D = \Delta_E = \Delta_F, \end{gathered} \tag{4.11}$$

i.e. the site parameters K'', Δ are the same for each site.

Conditions for model to be solvable

Consider a connected lattice $\mathscr{L}$ and extend the lines until they each cross some base line, as indicated by the broken lines in figure 1*a*. Assign site and angle coefficients to the sites and angles on the base line.

Every site A of $\mathscr{L}$ is then a vertex of a triangle consisting of the two lines through A and the base line, such as AGY in figure 1*a*. Call this the *basic triangle* with vertex A. In the next section we shall show that the eight-vertex model on $\mathscr{L}$ is exactly solvable (in the thermodynamic limit) provided the site and angle coefficients on the base line can be chosen to that the star-triangle conditions are satisfied for every basic triangle. If this can be done we say that the model is *Z-invariant*.

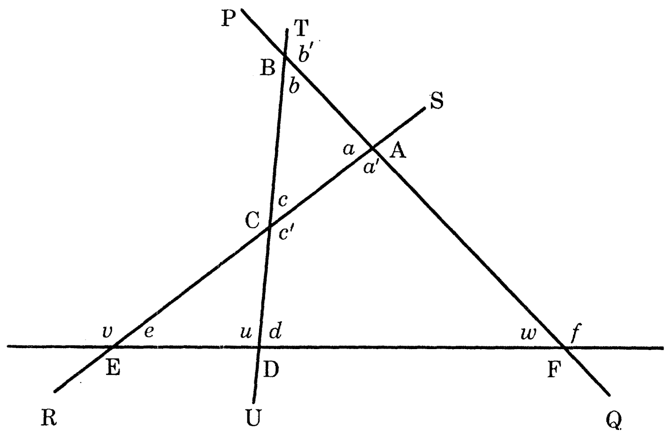

FIGURE 5. The quadrilateral ABCDEF. The lower-case letters denote the angle weights used in appendix A, e.g. $w = \exp\,(1K_{\mathrm{EFB}})$.

From (4.9), this implies that K'' must have the same value for all sites of $\mathscr{L}$ and all sites on the base line. So must Δ. Also, any triangle in $\mathscr{L}$ (not necessarily a face of $\mathscr{L}$) is part of a quadrilateral, with fourth side the base line. The other three triangles in this quadrilateral are basic, so from the quadrilateral theorem the star-triangle conditions must be satisfied *for every triangle in* $\mathscr{L}$. For example, they must be satisfied for the triangles ADE in figure 1*a* and *b*.

It follows that equation (4.3) is satisfied whenever a line of $\mathscr{L}$ is shifted across a vertex. Hence Z is unchanged by shifting the lines, so long as their order at the boundary is preserved. The correlation $\langle \sigma_l \sigma_m \rangle$ is also unchanged, provided no line is shifted across face l or face m.

Such models certainly exist. They can be constructed by choosing a set of site and angle coefficients for the base line, such that K'' and Δ have the same value at each such site. As is explicity shown in equation (A 3)–(A 7) of appendix A, the star-triangle conditions then determine the coefficients at the third vertex of every basic triangle, i.e. at every site of $\mathscr{L}$.

Since K'' and Δ are fixed, there is only one degree of freedom in choosing the angle coefficients of a site formed by the intersection of a lattice line with the base line. Ift here are N lattice lines, it follows that there are $N+2$ disposable parameters in the model, namely K'', Δ and N 'line parameters'.

Extended lattices

A lattice $\mathscr{L}$ can be extended either by moving the convex boundary outwards and extending the lattice lines to the new boundary, or by adding new lines, or both. If the model on $\mathscr{L}$ is Z-invariant, then the coefficients of the new sites and angles can be chosen so that the model on the extended lattice $\mathscr{L}'$ is also Z-invariant.

To see this, first consider a new site formed by extending two lines of $\mathscr{L}$ until they intersect. This new site is a vertex of a basic triangle, so its coefficients can be obtained from those of the sites on the base line by the star-triangle relations, and the Z-invariance condition remains satisfied.

If a new line is added, select an intersection A of this with a previous line. Assign site and angle coefficients to A such that K'', Δ have the same values as on the previous sites of $\mathscr{L}$.

Extend the new line to cross the base line at a point P. Let Q be the intersection of the old line through A with the base line. From figure 6 it is apparent that PQA is a basic triangle. The coefficients at P can be obtained from the star-triangle relations for PQA.

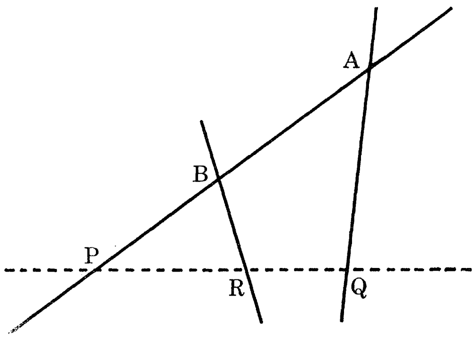

Figure 6. Assignment of coefficients to a new line ABP in $\mathscr{L}$: PRQ is the base line, AQ and BR are already existing lines. The coefficients at Q, R are given. If those of A are also given, then those at P, B are determined by the star-triangle relations for PQA, PRB.

Every other site B on the new line is a vertex of a basic triangle, for example BPR in figure 6. The coefficients at B can therefore be obtained from the star-triangle relations for BPR, thereby maintaining Z-invariance.

Note that if such a line crosses all previous lines of $\mathscr{L}$, it can be regarded as an alternative base line. From the quadrilateral theorem, the star-triangle conditions are satisfied for all triangles, in particular for those with the new line as a side. Hence if a lattice model is Z-invariant with respect to one base line, it is also Z-invariant with respect to any other.

5. Local thermodynamic properties

Consider a site A near the centre of $\mathscr{L}$. Let PAQ, RAS be the two lines through A, as in figure 1. Let l, m, n, p be the four faces round A, as in figure 2.

Extend $\mathscr{L}$ as follows: draw $2M$ lines parallel to PAQ, M of them being close together on one side of $\mathscr{L}$, the other M being close together on the other side. Similarly, draw $2M$ lines parallel to RAS.

This creates a parallelogram 'frame' for $\mathscr{L}$, each side being made up of M lines close together.

Suppose that no other line of $\mathscr{L}$ is parallel to either PAQ or RAS (if one is, rotate it slightly). Extend every such line to cross all the $4M$ framing lines. Move the convex boundary outwards to enclose all these intersections, and extend all lines to the boundary.

This creates an extended lattice $\mathscr{L}'$, as in figure 7. Assign coefficients to the new sites and angles in $\mathscr{L}'$ according to the rules given above, choosing the angle coefficients between PAQ (and RAS) and the framing lines to be K_{PAR} and K_{RAQ}, necessarily equal angles having equal coefficients. Then by considering triangles it is straightforward to verify that the angle coefficients for any intersection of two framing lines are also K_{PAR} and K_{RAQ}, necessarily equal angles having equal coefficients.

Suppose that originally the boundary spins in $\mathscr{L}$ (i.e. those on faces adjacent to the boundary) were fixed to be up. Do the same for $\mathscr{L}'$ and consider correlations between the spins round A, such as $\langle\sigma_l\rangle$, $\langle\sigma_l\sigma_n\rangle$.

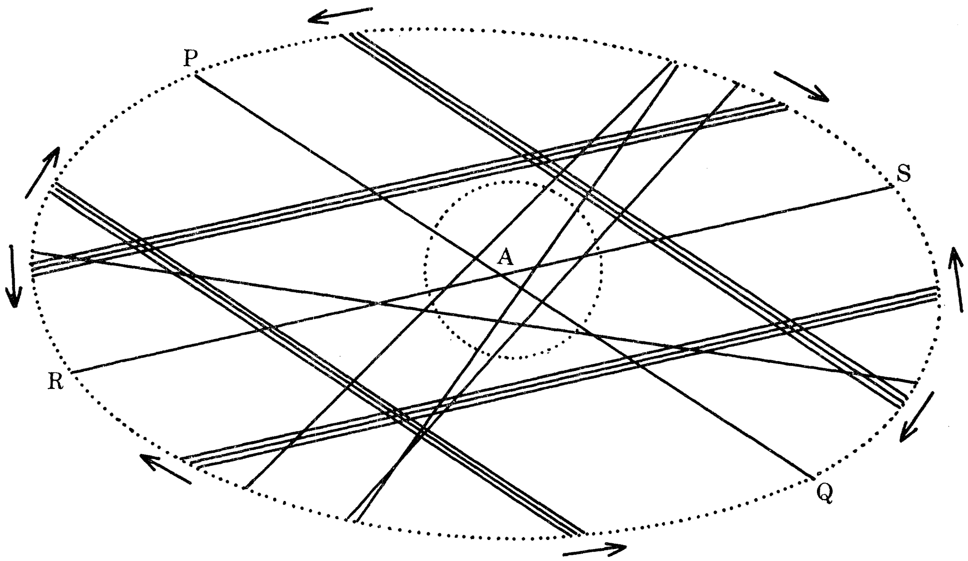

Figure 7. The lattice of figure 1*a*, extended as in §5. The inner dotted line is the old boundary, the outer one is the new boundary. The correlations of the four spins round A are unchanged by making parallel shifts of the framing lines inwards towards PQ and RS.

These are of course *not* the same for the model defined on $\mathscr{L}$ as for the model defined on $\mathscr{L}'$. However, if A is originally deep in the centre of $\mathscr{L}$ (i.e. any path through faces from A to the boundary crosses a large number of edges), then we expect the local correlations to be insensitive to the position of the boundary, and in the limit of $\mathscr{L}$ large we do expect them to be the same for $\mathscr{L}$ and $\mathscr{L}'$.

There is a problem here: if the Boltzmann weights are not all positive then local correlations may be sensitive to boundary conditions. From the construction used to extend $\mathscr{L}$ it is not obvious that all the new weights generated will be positive. However, this is probably not a serious difficulty. In particular, if $\mathscr{L}$ originally has a periodic structure (e.g. the Kagomé lattice discussed later), no new weights are generated by extending $\mathscr{L}$ to $\mathscr{L}'$, so if they were originally all positive for $\mathscr{L}$, then they are also all positive for $\mathscr{L}'$.

Now shift the framing lines of $\mathscr{L}'$ inwards towards A, keeping them parallel, until all the original sites of $\mathscr{L}$ (other than A) lie outside the frame. This does not alter the order of the lines at the boundary, and no line crosses A, so $\langle\sigma_l\rangle$, $\langle\sigma_l\sigma_m\rangle$, etc. are left unchanged, because of the Z-invariance of $\mathscr{L}'$.

Now, however, the picture is quite different: A is at the centre of a regular parallelogram lattice of $(2M+1)$ by $(2M+1)$ lines. Outside this lies the original lattice $\mathscr{L}$. We still expect the local correlations to be insensitive to what is going on many sites away from A, so if M is large we expect the local correlations to be the same as those of this regular lattice.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

But this lattice is just the regular square lattice eight-vertex model, with interaction coefficients K_{PAR}, K_{RAQ}, K''. Thus the correlations $\langle\sigma_l\rangle$, $\langle\sigma_l\sigma_n\rangle$, etc. *are the same as those of this regular model.*

In particular, the local magnetization, polarization and internal energy U_{A} of the lattice $\mathscr{L}$ are given by

$$\langle\sigma_l\rangle \quad = M(\Delta, K'') \tag{5.1}$$

$$\langle\sigma_l\sigma_m\rangle = P(\Delta, K'') \tag{5.2}$$

$$-\beta U_{\mathrm{A}} = K_{\mathrm{PAR}}\langle\sigma_l\sigma_n\rangle + K_{\mathrm{RAQ}}\langle\sigma_m\sigma_p\rangle + K''\langle\sigma_l\sigma_m\sigma_n\sigma_p\rangle = u(K_{\mathrm{PAR}}, K_{\mathrm{RAQ}}, K''), \tag{5.3}$$

where M, P, $-\beta^{-1}u$ are the magnetization, polarization and internal energy per site of the regular square-lattice eight-vertex model with interaction coefficients K_{PAR}, K_{RAQ}, K''. The first two of these are known to depend on K_{PAR}, K_{RAQ} only via Δ (Barber & Baxter 1973; Baxter & Kelland 1974). (This can also be established by the present methods.)

Note that $\langle\sigma_l\rangle$ is therefore the same (in the thermodynamic limit) for every face of $\mathscr{L}$. Similarly, $\langle\sigma_l\sigma_m\rangle$ is the same for every edge.

Free energy

Let $F = -\beta^{-1}\ln Z$ be the total free energy of $\mathscr{L}$. Increment all the site and angle coefficients by infinitesimal amounts $\delta K''$, δK_{PAR}, etc. Then from (2.1), (2.2) and (2.3) the increment induced in $-\beta F$ is

$$-\delta(\beta F) = \Sigma[\delta K_{\mathrm{PAR}}\langle\sigma_l\sigma_n\rangle + \delta K_{\mathrm{RAQ}}\langle\sigma_m\sigma_p\rangle + \delta K''\langle\sigma_l\sigma_m\sigma_n\sigma_p\rangle]. \tag{5.4}$$

where the summation is over all sites A of $\mathscr{L}$, as in (1).

Let $f_{\mathrm{A}} = f(K_{\mathrm{PAR}}, K_{\mathrm{RAQ}}, K'')$ be the free energy per site of the regular square lattice model, with coefficients K_{PAR}, K_{RAQ}, K''. Then from the above remarks

$$\left.\begin{aligned}\langle\sigma_l\sigma_n\rangle &= -\partial(\beta f_{\mathrm{A}})/\partial K_{\mathrm{PAR}},\\ \langle\sigma_m\sigma_p\rangle &= -\partial(\beta f_{\mathrm{A}})/\partial K_{\mathrm{RAQ}},\\ \langle\sigma_l\sigma_m\sigma_n\sigma_p\rangle &= -\partial(\beta f_{\mathrm{A}})/\partial K'',\end{aligned}\right\} \tag{5.5}$$

provided A is deep inside $\mathscr{L}$.

Assuming that we can ignore the contribution to the sum in (5.4) of sites that are near the boundary, it follows that

$$\delta(\beta F) = \Sigma\,\delta(\beta f_{\mathrm{A}}), \tag{5.6}$$

where $\delta(\beta f_{\mathrm{A}})$ is the increment induced in βf_{A}. Thus

$$\beta[F - \Sigma f_{\mathrm{A}}]$$

is stationary with respect to variations in the interaction coefficients, provided the model on $\mathscr{L}$ is Z-invariant.

However, while keeping the model Z-invariant, one can continuously vary the interactions until all the coefficients are large and positive, when it is trivially true that $F - \Sigma f_{\mathrm{A}}$ is zero. Thus for any Z-invariant model one must have

$$F = \Sigma f(K_{\mathrm{PAR}}, K_{\mathrm{RAQ}}, K''), \tag{5.7}$$

the summation being over all sites A of $\mathscr{L}$.

This is the key result of this paper. The free energy of any Z-invariant lattice model is the sum of site free energies, the site energies being those of the regular square lattice model.

There are previous results that have suggested this: notably an inhomogeneous square lattice model (Baxter 1972, eqn 10.4) and the six-vertex model on the Kagomé lattice (Baxter, Temperley & Ashley 1977, eqn 47).

6. Elliptic function parametrization

The star-triangle relations and the Z-invariance conditions can be written very simply by introducing elliptic functions.

The four-spin interaction coefficient K'' must have the same value at each site of $\mathscr{L}$, and so must Δ. If these two values are given, then (4.8) is a relation between the two angle coefficients K_j and K_j' at site j.

Define two site-independent parameters ω, Ω by

$$\coth\ 2K'' = \cosh\Omega$$

$$-\Delta\coth 2K'' = \cosh\omega. \tag{6.1}$$

Then (4.8) can be written as

$$\cosh\omega = \cosh 2K_j \cosh 2K_j' + \cosh\Omega \sinh 2K_j \sinh 2K_j'. \tag{6.2}$$

Although this relation concerns only a single site of $\mathscr{L}$, formally it is the same as that between the sides ω, $2K_j$, $2K_j'$ of an hyperbolic triangle, with angle $\pi + \mathrm{i}\Omega$ between the sides $2K_j$, $2K_j'$ (Onsager 1944, p. 135; Coxeter 1947). This is the same as that for a spherical triangle with sides of pure imaginary length.

It is well known that this relation can be simplified by introducing elliptic functions of modulus

$$k = \sinh\Omega/\sinh\omega \tag{6.3}$$

(Greenhill 1892, §129). Onsager (1944, p. 144) refers to this as a uniformizing substitution.

From (6.1) and (6.3) it follows that

$$k^{-2} = \Delta^2\cosh^2 2K'' - \sinh^2 2K'', \tag{6.4}$$

and from (4.8) that

$$k^{-2} = \frac{16(1+vv'v'')\,(v+v'v'')\,(v'+v''v)\,(v''+vv')}{(1-v^2)^2(1-v'^2)^2(1-v''^2)}, \tag{6.5}$$

where

$$v = \tanh K_j, \quad v' = \tanh K_j', \quad v'' = \tanh K''. \tag{6.6}$$

It is interesting, but probably quite irrelevant, to note that this k is the same as the elliptic modulus which occurs in the solution of a triangular two-spin Ising model with interaction coefficients K_j, K_j', K'' (Green 1963; Stephenson 1964).

Let $\mathscr{K}$, $\mathscr{K}'$ be the complete elliptic integrals of the first kind of moduli, k, $k' = (1-k^2)^{\frac{1}{2}}$, respectively. They are *not* to be confused with any of the interaction coefficients used above, notably with K_j and K_j'.

Following Greenhill (1892), but making a few notational changes, we define α_j, β_j, λ by

$$\left.\begin{aligned} 2K_j' &= -\mathrm{i}\,\mathrm{am}\,[\mathrm{i}(\mathscr{K}'-\alpha_j)], \\ 2K_j &= -\mathrm{i}\,\mathrm{am}\,[\mathrm{i}(\mathscr{K}'-\beta_j)], \\ 2K'' &= \mathrm{i}\,\mathrm{am}\,[\mathrm{i}(\mathscr{K}'-\lambda)]. \end{aligned}\right\} \tag{6.7}$$

Then from (6.1), (6.3) and various standard relations for the Jacobi elliptic functions (Gradshteyn & Ryzhik 1965, §§ 8.143 and 8.151.2)

$$\left.\begin{aligned} \cosh 2K'_j &= \mathrm{i}k^{-1}\,\mathrm{ds}\,(\mathrm{i}\alpha_j),\\ \sinh 2K'_j &= \mathrm{i}/[k\,\mathrm{sn}\,(\mathrm{i}\alpha_j)],\\ \cosh 2K_j &= \mathrm{i}k^{-1}\,\mathrm{ds}\,(\mathrm{i}\beta_j),\\ \sinh 2K_j &= \mathrm{i}/[k\,\mathrm{sn}\,(\mathrm{i}\beta_j)],\\ \coth 2K'' &= \cosh\Omega = -\mathrm{dn}\,(\mathrm{i}\lambda),\\ \sinh 2K'' &= -\mathrm{i}/[k\,\mathrm{sn}\,(\mathrm{i}\lambda)],\\ \omega &= \mathrm{i}\,\mathrm{am}\,[\mathrm{i}(2\mathscr{K}'-\lambda)],\\ \Delta &= -\mathrm{cn}(\mathrm{i}\lambda)/\mathrm{dn}(\mathrm{i}\lambda), \end{aligned}\right\} \tag{6.8}$$

where $\mathrm{ds}(u) = \mathrm{dn}(u)/\mathrm{sn}(u)$.

Substituting these expressions into (6.2) and using the elliptic function identity

$$k^2\,\mathrm{cn}\,(u+v)\,\mathrm{sn}\,u\,\mathrm{sn}\,v = \mathrm{dn}\,u\,\mathrm{dn}\,v - \mathrm{dn}\,(u+v), \tag{6.9}$$

we find that (4.8) is satisfied if

$$\alpha_j + \beta_j = \lambda. \tag{6.10}$$

Also, by using the relations (6.7) and (6.8) in the equation (4.10), and using (6.9) and (6.10) to simplify the right hand side, (4.10) becomes

$$[\mathrm{dn}\,(\mathrm{i}\beta_2)\,\mathrm{dn}\,(\mathrm{i}\beta_3) - \mathrm{dn}\,(\mathrm{i}\alpha_1)]/[k^2\,\mathrm{sn}\,(\mathrm{i}\beta_2)\,\mathrm{sn}\,(i\beta_3)] = \mathrm{cn}\,(\mathrm{i}\alpha_1). \tag{6.11}$$

From (6.9) this is clearly satisfied if

$$\alpha_1 = \beta_2 + \beta_3 \tag{6.12}$$

or, by using (6.10), if

$$\alpha_1 + \alpha_2 + \alpha_3 = 2\lambda, \quad \beta_1 + \beta_2 + \beta_3 = \lambda. \tag{6.13}$$

Since these relations are unchanged by permuting the suffixes 1, 2, 3, the other two equations that can thereby be obtained from (4.10) are also satisfied.

The relations (4.7) and (4.10) imply (4.4), so the original star-triangle relations are satisfied by the definitions (6.4) and (6.7), and the relations (6.10) and (6.13).

So far we have made no restriction on the values of the interaction coefficients, other than those imposed by the star-triangle relations. From (6.4), k may be greater or less than one, or may be imaginary. Whatever the value of k, there will always be parameters α_j, β_j, λ that satisfy (6.7), (6.10) and (6.13), but they may be complex.

To fix our ideas, and to give a real single-valued definition of α_j, β_j, λ, it is desirable to focus attention on the case when at every site j of $\mathscr{L}$ the two angle coefficients K_j and K'_j satisfy

$$\sinh\,(K_j + K'_j) > \mathrm{e}^{-2K''}\cosh\,(K_j - K'_j). \tag{6.14}$$

From (3.5) this is equivalent to the condition $c_j > a_j + b_j + d_j$. This is the ferromagnetically ordered phase of the eight-vertex model.

From (4.8), this implies that

$$\Delta < -1, \tag{6.15}$$

so from (6.4), $k^{-2} > 1$ and we can choose k so that

$$0 < k < 1. \tag{6.16}$$

The obvious solution of (6.7), together with the definition of $\mathscr{K}$, $\mathscr{K}'$, is then

$$\alpha_j = \int_{2K_j'}^{\infty} (1+k^2\sinh^2\phi)^{-\frac{1}{2}}\,d\phi, \tag{6.17a}$$

$$\beta_j = \int_{2K_j}^{\infty} (1+k^2\sinh^2\phi)^{-\frac{1}{2}}\,d\phi, \tag{6.17b}$$

$$\lambda = \int_{-2K''}^{\infty} (1+k^2\sinh^2\phi)^{-\frac{1}{2}}\,d\phi, \tag{6.17c}$$

$$\mathscr{K}' = \int_{0}^{\infty} (1+k^2\sinh^2\phi)^{-\frac{1}{2}}\,d\phi, \tag{6.17d}$$

$$\mathscr{K} = \int_{0}^{\frac{1}{2}\pi} (1-k^2\sin^2\theta)^{-\frac{1}{2}}\,d\theta, \tag{6.17e}$$

so that α_j, β_j, λ are real and positive. For this case this solution is indeed the one that satisfies (6.10) and (6.13), so

$$0 < \alpha_j, \beta_j < \lambda < 2\mathscr{K}'. \tag{6.18}$$

Note that K_1', K_2', K_3', α_1, α_2, α_3 are associated with the exterior angles of the triangle ABC, while K_1, K_2, K_3, β_1, β_2, β_3 are associated with the interior angles.

Z-invariance conditions in terms of the elliptic angle parameters

In the above equations (6.2) to (6.18) we have considered a triangle with vertices 1, 2, 3, interior angle coefficients K_1, K_2, K_3, and exterior angle coefficients K_1', K_2', K_3'. To develop a notation appropriate to the whole lattice, we note that K'', Δ, k, λ are the same for all sites, to an angle PAR is assigned an interaction coefficient K_{PAR}, and, from (6.17*a* or *b*), an elliptic angle parameter

$$\alpha_{\mathrm{PAR}} = \int_{2K_{\mathrm{PAR}}}^{\infty} (1+k^2\sinh^2\phi)^{-\frac{1}{2}}\,d\phi. \tag{6.19}$$

Then a lattice $\mathscr{L}$ is Z-invariant if the relations (6.10) and (6.13) are satisfied for every basic triangle formed by two lattice lines and the base line, i.e. if

(i) the sum of the two elliptic angle parameters at every site is λ,

(ii) the sum of the elliptic parameters of the interior (exterior) angles of every basic triangle is λ (2λ).

General formulae for all the elliptic angle parameters can now be given. Suppose that all sites of $\mathscr{L}$ lie on the same side of the base line and rotate the lattice until the base line is horizontal and below $\mathscr{L}$, as in figure 1. Label the lattice lines 1, ..., N. Let X be a point on the base line to the right of all the lattice lines.

Consider a line r. Let A be a lattice site on r and E its intersection with the base line, as in figure 5 (regarding EDFX as the base line). Define

$$\alpha_{\mathrm{r}} = \alpha_{\mathrm{AEX}}. \tag{6.20}$$

There are N such 'line angle parameters'.

Now consider a typical site of $\mathscr{L}$, such as A in figure 5. Let r be the line AE, s be the line AF, E and F lying on the base line, E to the left of F.

Since AEF is a basic triangle, it follows from (i) and (ii) that

$$\left.\begin{aligned}\alpha_{\mathrm{EAF}} &= \alpha_{\mathrm{s}}-\alpha_{\mathrm{r}},\\ \alpha_{\mathrm{EAB}} &= \lambda+\alpha_{\mathrm{r}}-\alpha_{\mathrm{s}}.\end{aligned}\right\} \tag{6.21}$$

Thus the elliptic parameters of the angles of intersection of lines r and s can be simply written in terms of $\alpha_{\mathrm{s}}-\alpha_{\mathrm{r}}$ and λ.

The quadrilateral theorem is now trivial. Let t be the line BCD in figure 5. Then the star-triangle relations (i) and (ii) for AEF, BFD, CDE imply (6.21), i.e.

$$\left.\begin{aligned}\alpha_{\mathrm{BAC}} &= \lambda+\alpha_{\mathrm{r}}-\alpha_{\mathrm{s}},\\ \alpha_{\mathrm{ACB}} &= \alpha_{\mathrm{t}}-\alpha_{\mathrm{r}},\\ \alpha_{\mathrm{CBA}} &= \alpha_{\mathrm{s}}-\alpha_{\mathrm{t}}.\end{aligned}\right\} \tag{6.22}$$

Adding these equations gives the required star-triangle relation for ABC, namely

$$\alpha_{\mathrm{BAC}}+\alpha_{\mathrm{ACB}}+\alpha_{\mathrm{CBA}} = \lambda. \tag{6.23}$$

Thus the sum of the elliptic parameters of the interior angles must be 2λ.

More generally, one can readily prove:

(iii) the sum of the elliptic parameters of the exterior angles of any polygon in $\mathscr{L}$ must be 2λ.

This is a useful observation, since if (i) and (iii) are satisfied, then the model on $\mathscr{L}$ is Z-invariant. This provides an alternative definition of Z-invariance that does not require the artificial introduction of a base line.

Geometric model

This formulation with elliptic functions and integrals makes it quite clear that the Z-invariance conditions have a geometric interpretation. In fact, for a given lattice $\mathscr{L}$ and given values of $k, \lambda(K'', \Delta)$ there is a particularly obvious choice of the angle parameters, namely that for any angle PAR

$$\alpha_{\mathrm{PAR}} = (\lambda/\pi)\times\text{the angle PAR (radians)}. \tag{6.24}$$

The conditions (i), (ii), (iii) are then automatically satisfied. We call this the 'geometric' model. Many models, notably the anisotropic square and Kagomé lattice models, can be converted to geometric models by rotating some of the lattice lines (e.g. so as to convert the square lattice into a parallelogram lattice).

The condition (6.14) can be somewhat relaxed without introducing complex elliptic angle parameters. It can be replaced by the requirement that the v, v', v'' defined by (6.6) satisfy

$$v+v'v'', \quad v'+v''v, \quad v''+vv' > 0 \tag{6.25}$$

for every site of $\mathscr{L}$. This is automatically satisfied if the model is ferromagnetic, i.e. all the interaction coefficients are positive. From (6.5) it implies only that $k^2 > 0$, whereas (6.14) implies $0 < k^2 < 1$. Real positive parameters λ, α_{PAR} may still be defined by (6.17*c*) and (6.19), and the Z-invariance conditions are still equivalent to the conditions (i) and (iii).

Alternatively, if (6.25) is satisfied but (6.14) is not, then $k^2 > 1$ and it is natural to use elliptic functions of modulus $k^* = k^{-1}$. Specific formulae for doing this are given in equations (9.3)–(9.8).

Connection with Onsager's parametrization: the case $K'' = 0$

As was remarked in section 2, if $K'' = 0$ the eight-vertex model factors into two independent two-spin Ising models. From (4.8) and (6.4),

$$k = 1/|\sinh 2K_j \sinh 2K_j'|. \tag{6.26}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

For the square lattice this is the elliptic modulus used by Onsager for the low-temperature case (Onsager 1944, eqn 2.1*a*). Onsager's parameters a, $\mathscr{K}'-a$ are our parameters $\mathscr{K}'-\alpha_j$, $\mathscr{K}'-\beta_j$. From (6.17*c*)

$$\lambda = \mathscr{K}' \tag{6.27}$$

so from (6.10) the sum of $\mathscr{K}'-\alpha_j$ and $\mathscr{K}'-\beta_j$ is $\mathscr{K}'$, in agreement with Onsager.

7. Expressions for f, M, P

The regular square-lattice function f in (5.7) has been obtained (Baxter 1972), using the ferroelectric formulation of the model described in §3. The result can be summarized as follows (negating b in the 1972 paper).

Let K_j, K_j', K'' $(=K_j'')$ be the interaction coefficients (the same for every site in the regular square lattice model). Let a, b, c, d be the Boltzmann weights defined by (3.5) (temporarily dropping the suffix j). Define

$$\left.\begin{aligned} w_1 &= \tfrac{1}{2}(c+d), & w_2 &= \tfrac{1}{2}(c-d),\\ w_3 &= \tfrac{1}{2}(a+b), & w_4 &= \tfrac{1}{2}(a-b).\end{aligned}\right\} \tag{7.1}$$

Rearrange and negate (if necessary) $w_1, \ldots, w_4$ until they satisfy

$$w_1 > w_2 > w_3 > |w_4|. \tag{7.2}$$

Now define new weights a, b, c, d so that (7.1) is again satisfied. Call these a', b', c', d'. From (7.2) they are positive and satisfy

$$c' > a'+b'+d'. \tag{7.3}$$

(This procedure maps the model into the ordered ferromagnetic phase, while leaving the partition function unchanged.)

Define an elliptic modulus k_{I} and parameters η, v such that

$$\left.\begin{aligned} a':b':c':d' = {}& \mathrm{sn}\,(\eta+v, k_{\mathrm{I}}):\mathrm{sn}\,(\eta-v, k_{\mathrm{I}}):\\ & \mathrm{sn}\,(2\eta, k_{\mathrm{I}}): -k_{\mathrm{I}}\,\mathrm{sn}\,(2\eta, k_{\mathrm{I}})\,\mathrm{sn}\,(\eta+v, k_{\mathrm{I}})\,\mathrm{sn}\,(\eta-v, k_{\mathrm{I}}).\end{aligned}\right\} \tag{7.4}$$

These can and are to be chosen so that k_{I} is real, satisfying

$$0 < k_{\mathrm{I}} < 1, \tag{7.5}$$

while η and v are pure imaginary, satisfying

$$|\mathrm{Im}\,(v)| < \mathrm{Im}\,(\eta) < \tfrac{1}{2}\mathscr{K}_1', \tag{7.6}$$

where in this section $\mathscr{K}_{\mathrm{I}}$ and $\mathscr{K}_{\mathrm{I}}'$ are the complete elliptic integrals of the first kind of moduli k_{I}, $k_{\mathrm{I}}' = (1-k_{\mathrm{I}}^2)^{\frac{1}{2}}$, respectively. They are *not* interaction coefficients.

Define q, x, z by (Baxter 1972, eqns D4–D8)

$$q = \exp\,(-\pi\mathscr{K}_{\mathrm{I}}'/\mathscr{K}_{\mathrm{I}}), x = \exp\,(\mathrm{i}\pi\eta/\mathscr{K}_{\mathrm{I}}), z = \exp\,(\mathrm{i}\pi v/\mathscr{K}_{\mathrm{I}}). \tag{7.7}$$

Then q, x, z are positive real, satisfying

$$0 < q < x^2 < 1, \quad x < z < x^{-1}, \tag{7.8}$$

and f, the free energy per site, is given by (Baxter 1972, eqn D37)

$$-\beta f(K_j, K_j', K'') = \ln c' + \sum_{n=1}^{\infty} \frac{x^{-n}(x^{2n}-q^n)^2(x^n+x^{-n}-z^n-z^{-n})}{n(1-q^{2n})\,(1+x^{2n})}. \tag{7.9}$$

Also, Barber & Baxter (1973) and Baxter & Kelland (1974) conjecture that in the ordered phases M and P are given by

$$M = \prod_{n=1}^{\infty} \frac{1-x^{4n-2}}{1+x^{4n-2}} \tag{7.10}$$

$$P = \prod_{n=1}^{\infty} \left[\frac{1+q^n}{1-q^n}\frac{1-x^{2n}}{1+x^{2n}}\right]^2. \tag{7.11}$$

They are certainly independent of z.

Relation between the elliptic parametizations in the ordered ferromagnetic phase

From (3.5) and (4.8), K'' and Δ may be expressed as functions of a, b, c, d:

$$\exp(4K'') = cd/ab \tag{7.12}$$

$$\Delta = \tfrac{1}{2}(a^2+b^2-c^2-d^2)/(ab+cd). \tag{7.13}$$

Suppose that (6.14) is satisfied, i.e. that $c > a+b+d$. Then the a', b', c', d' in (7.4) are the original weights a, b, c, d. Substituting (7.4) into (7.12) and (7.13) gives

$$\exp(2K'') = -\mathrm{i}k_{\mathrm{I}}^{\frac{1}{2}}\,\mathrm{sn}\,(2\eta, k_{\mathrm{I}}) \tag{7.14}$$

$$\Delta = -\frac{\mathrm{cn}\,(2\eta,k_{\mathrm{I}})\,\mathrm{dn}\,(2\eta,k_{\mathrm{I}})}{1-k_{\mathrm{I}}\,\mathrm{sn}^2\,(2\eta,k_{\mathrm{I}})}. \tag{7.15}$$

Substituting these expressions into (6.4) gives

$$k = 2k_{\mathrm{I}}^{\frac{1}{2}}/(1+k_{\mathrm{I}}). \tag{7.16}$$

Thus the elliptic moduli k, k_{I} are related by a Landen transformation. By using §8.152 of Gradshteyn & Ryzhik (1965), it follows from (7.14) or (7.15) and (6.8) that

$$\mathrm{i}\lambda = (1+k_{\mathrm{I}})\,2\eta. \tag{7.17}$$

Also, from (3.5) $$\sinh 2K_j' = (bc-ad)/[2(abcd)^{\frac{1}{2}}], \tag{7.18}$$

so, by using (7.4) $$\sinh 2K_j' = \tfrac{1}{2}\mathrm{i}k_{\mathrm{I}}^{-\frac{1}{2}}[1+k_{\mathrm{I}}\,\mathrm{sn}^2(\eta+v,k_{\mathrm{I}})]/\mathrm{sn}\,(\eta+v,k_{\mathrm{I}}). \tag{7.19}$$

From §8.152 of Gradshteyn & Ryzhik (1965) and (6.8), it follows that

$$\mathrm{i}\alpha_j = (1+k_{\mathrm{I}})\,(\eta+v), \tag{7.20}$$

and, noting that negating v is equivalent to interchanging K_j and K_j',

$$\mathrm{i}\beta_j = (1+k_{\mathrm{I}})\,(\eta-v). \tag{7.21}$$

Note that (7.17), (7.20) and (7.21) imply the relation (6.10).

The elliptic integrals $\mathscr{K}$, $\mathscr{K}'$, $\mathscr{K}_{\mathrm{I}}$, $\mathscr{K}_{\mathrm{I}}'$ are related by

$$\mathscr{K} = (1+k_{\mathrm{I}})\,\mathscr{K}_{\mathrm{I}}, \quad \mathscr{K}' = \tfrac{1}{2}(1+k_{\mathrm{I}})\,\mathscr{K}_{\mathrm{I}}'. \tag{7.22}$$

By eliminating η, v, $\mathscr{K}_{\mathrm{I}}$, $\mathscr{K}_{\mathrm{I}}'$ between (7.7), (7.17) and (7.20)–(7.22), and exhibiting the site dependence of z, it follows that

$$q = \exp(-2\pi\mathscr{K}'/\mathscr{K}_{\mathrm{I}}), \quad x = \exp(-\pi\lambda/2\mathscr{K}),$$

$$z = z_j = \exp[-\pi(\alpha_j-\beta_j)/2\mathscr{K}]. \tag{7.23}$$

Together with (6.17), (6.4) and (4.8), this provides an explicit real definition of q, x, z_j for the ordered ferromagnetic phase.

Note that q and x depend only on k and λ, i.e. on K'' and Δ. Thus they are the same for all sites of the lattice $\mathscr{L}$, while z varies from site to site. Its value z_j at site j is given by (7.23), α_j and β_j being the elliptic parameters of the two angles at j. The order of the two angles is irrelevant here, since (7.9) is unchanged by inverting z.

Phase boundaries

The free energy function $f(K_j, K_j', K'')$ defined by (3.5) and (7.1)–(7.9) is analytic except when the middle two w's, in numerically decreasing order, are equal, i.e. when

$$a = b+c+d, b = a+c+d, c = a+b+d \quad \text{or} \quad d = a+b+c. \tag{7.24}$$

At these surfaces the correlation length goes to infinity (Johnson, Krinsky & McCoy 1973), so they are surfaces of critical points.

From (7.12), (7.13), (6.4) and (3.7),

$$k^{-2} = a^*b^*c^*d^*/abcd, \tag{7.25}$$

$$\frac{1-k^2}{k^2} = \frac{(a-b+c+d)(-a+b+c+d)(a+b+c-d))(-a-b+c-d)}{16\,abcd}. \tag{7.26}$$

Thus $k^2 = 1$ if, and only if, the system is critical.

The system is in an ordered phase if one of a, b, c, d is greater than the sum of the other three. From (7.26) and (6.4) this implies that

$$0 < k^2 < 1(|\Delta| > 1), \tag{7.27}$$

and vice versa.

The system is disordered if each of a, b, c, d is less than the sum of the other three, i.e. if

$$k^{-2} < 1\,(|\Delta| < 1). \tag{7.28}$$

Although a, b, c, d may vary from site to site, k (and Δ) does not. Thus for any lattice $\mathscr{L}$ we expect a Z-invariant model to be ordered if (7.27) is satisfied, disordered if (7.28) is satisfied, and to be critical if $k^2 = 1$ (and $\Delta = \pm 1$).

The definition (7.1)–(7.9) of f is somewhat cumbersome to use in the disordered phase, there being eight different cases to consider. This is related to the fact that k^2 can be either positive (greater than one), or negative. Nevertheless, f is analytic throughout the disordered phase.

8. Kagomé lattice eight-vertex model

So far, we have considered a very general 'lattice' $\mathscr{L}$, made up of almost any set of intersecting straight lines. The advantage of doing this is that it brings out very clearly the trigonometric character of the star-triangle conditions on the interaction coefficients of the eight-vertex model. It is, however, unnecessarily general for the purpose outlined in §1.

In this section we specialize to a regular eight-vertex model on the Kagomé lattice shown in figure 8. This can be divided into three equivalent sub-lattices, labelled 1, 2 and 3 in figure 8.

Associate spins with the *faces* of the lattice. On every up-triangle assign the same set of two-spin interaction coefficients K_1, K_2, K_3, K_1', K_2', K_3' as indicated in figure 9. Also assign the same four-spin interaction coefficient K'' to every set of four spins round a site of the Kagomé lattice. The

Hamiltonian is then given by (2.1), with K_{PAR} and K_{RAQ} equal to the appropriate two-spin coefficient, and $K''_A = K''$.

The spins on the triangular faces form a honeycomb lattice, interacting with their nearest neighbours with coefficients K_1, K_2, K_3, as indicated in figure 9. The spins on the hexagonal faces form a triangular lattice, with interaction coefficients K'_1, K'_2, K'_3.

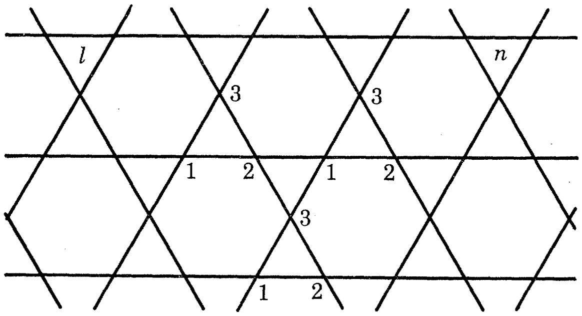

Figure 8. The Kagomé lattice, with sites divided into three equivalent classes 1, 2, 3.

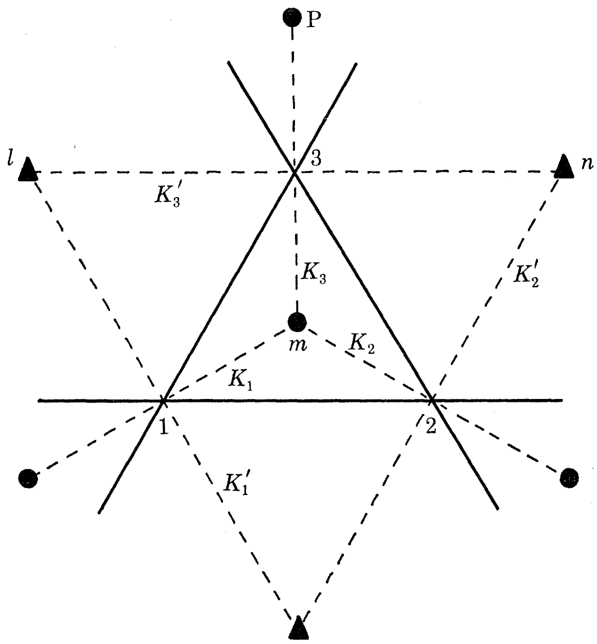

Figure 9. An up-triangle of the Kagomé lattice. The solid lines are lattice edges, while the circles and triangles denote the spins associated with the lattice faces. Broken lines represent two-spin intersections, the corresponding coefficients $K_1, \ldots, K'_3$ being shown.

Thus this eight-vertex model consists of an honeycomb and a triangular Ising model, with four-spin interactions between them.

We require that the star-triangle relations (4.4) be satisfied for every up-triangle of the type shown in figure 9, i.e.

$$\left.\begin{aligned} \exp(2K'_2+2K'_3) &= \cosh(K_1+K_2+K_3-K'')/\cosh(-K_1+K_2+K_3+K''),\\ \exp(2K'_3+2K'_1) &= \cosh(K_1+K_2+K_3-K'')/\cosh(K_1-K_2+K_3+K''),\\ \exp(2K'_1+2K'_2) &= \cosh(K_1+K_2+K_3-K'')/\cosh(K_1+K_2-K_3+K''). \end{aligned}\right\} \tag{8.1}$$

The model is then Z-invariant, so the thermodynamic properties are given by (5.1), (5.2) and (5.7). In particular, the mean free energy per site of the Kagomé lattice model is given by

$$f_{\text{Kagomé}} = \tfrac{1}{3}[f(K_1, K_1', K'') + f(K_2, K_2', K'') + f(K_3, K_3', K'')], \tag{8.2}$$

where $f(K_j, K_j', K'')$ is the free energy per site of a regular square lattice model with coefficients K_j, K_j', K'', given by equations (3.5) and (7.1)–(7.9).

Since the two-spin interaction coefficients K_1, K_2, K_3, K_1', K_2', K_3' are arranged in the same way in figure 9 as in figure 4, many of the formulae of previous sections, e.g. (4.10), can be applied directly to this Kagomé lattice model. Note however that there is a difference in viewpoint: previously $K_1, \ldots, K_3'$ were the coefficients of some typical triangle in $\mathscr{L}$, different for different triangles. Here $K_1, \ldots, K_3'$ (and K'') specify the complete Hamiltonian, being the same for all up-triangles in the Kagomé lattice.

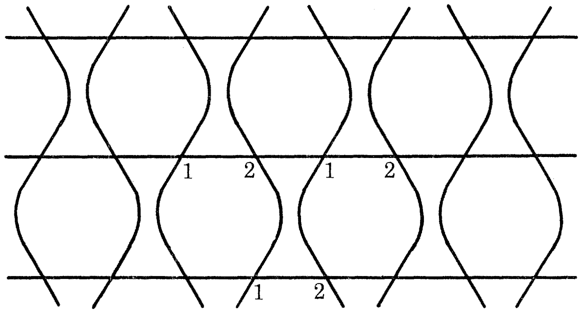

FIGURE 10. The deformation of the Kagomé lattice that corresponds to letting K_3 become infinite. The resulting lattice is essentially rectangular.

Previous models as special cases

In general the conditions (8.1) are temperature-dependent. Nevertheless, the model is still very interesting, since it contains as special cases all the previously solved models (i) to (vi) listed in §1.

If $K'' = 0$, the model factors into a honeycomb and a triangular Ising model. The three coefficients K_1, K_2, K_3 of the honeycomb model may be chosen arbitrarily. The coefficients K_1', K_2', K_3' of the triangular model are then given by (8.1), but this is the star-triangle relation, so the two models are equivalent and the properties of either can be deduced from the properties of their product. Thus this model includes (ii), and hence (i), as a special case.

Alternatively, suppose $K'' \neq 0$ but let $K_3 \to +\infty$. Then from (8.1)

$$K_1' = K_2', \quad K_2' = K_1, \tag{8.3a}$$

$$K_3' = -K''. \tag{8.3b}$$

Consider the interactions between the four spins on faces l, m, n, p round a site of type 3, as in figure 9. Since $K_3 \to +\infty$, σ_m and σ_p must be equal. The remaining interactions are

$$K_3' \sigma_l \sigma_n + K'' \sigma_l \sigma_n \sigma_m \sigma_p. \tag{8.4}$$

Since $\sigma_m = \sigma_p = \pm 1$, it follows that $\sigma_m \sigma_p = 1$. From (8.3b) the interaction (8.4) therefore vanishes.

Thus in the limit $K_3' \to \infty$ the faces m and p can be identified and the faces l and n separated. Graphically this is equivalent to deforming the Kagomé lattice of figure 8 to the lattice of figure 10.

The latter is simply a square lattice. Using (8.3a) we find that the model is now a regular square lattice eight-vertex model, with two-spin interactions K_1 and K_2, and four-spin interaction K''. Thus (iv), and hence (iii), are special cases of the Kagomé lattice model.

An interesting isotropic case is when $K_1 = K_2 = K_3 = K''$, which from (8.1) implies that $K_1' = K_2' = K_3' = 0$. Let $\sigma_1, \ldots, \sigma_6$ be the six spins round an up-triangle, as in figure 4a. Then from (4.2a) the contribution of the triangle to the partition function is

$$W_1 = 2\cosh\left[K''(\sigma_1+\sigma_3+\sigma_5+\sigma_1\sigma_6\sigma_2+\sigma_3\sigma_2\sigma_4+\sigma_5\sigma_4\sigma_6)\right]. \tag{8.5}$$

Using $\sigma_i^2 = 1$, one can verify that this is the same as

$$W_1 = 2\cosh\left[K''(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_5+\sigma_5\sigma_6+\sigma_6\sigma_1)\right]. \tag{8.6}$$

(An easy way to do this is to verify that the squares of the bracketted expressions in (8.5) and (8.6) are the same.)

From (8.6), W_1 can be written

$$W_1 = \sum_{\sigma_0} \exp\left[K''\sigma_0(\sigma_1\sigma_2+\ldots+\sigma_6\sigma_1)\right], \tag{8.7}$$

where σ_0 can be regarded as the spin inside the up-triangle, as in figure 4a. But this is just the triangle contribution to the partition function of a system with Hamiltonian

$$-\beta\mathscr{H} = K''\Sigma\,\sigma_i\sigma_j\sigma_k, \tag{8.8}$$

the summation being over all triplets of spins consisting of one spin inside an up-triangle and two surrounding spins that are adjacent to one another. If two spins are regarded as 'neighbours' if they both lie in such a triplet, then they form a triangular lattice, and the sum in (8.8) is over all faces of this lattice. Thus (8.8) is then the Hamiltonian of the three-spin model (v), which is therefore also a special case of the Kagomé lattice eight-vertex model.

As was remarked in §1, (v) can also be transformed to a square-lattice eight-vertex model (Baxter & Enting 1976).

Finally, by using the vertex formulation of §3 and remembering that K'' is site-independent, (3.5) and (8.1) imply that the vertex weights must satisfy

$$c_1 d_1/a_1 b_1 = c_2 d_2/a_2 b_2 = c_3 d_3/a_3 b_3, \tag{8.9}$$

$$b_i/c_i = (a_j a_k - b_j b_k)/(c_j c_k - d_j d_k), \tag{8.10}$$

for all permutations (i, j, k) of (1, 2, 3).

Setting $d_1 = d_2 = d_3 = 0$, the model becomes the six-vertex model solved by Baxter, Temperley & Ashley (1977), (8.10) becoming their restriction (6).

Classification of phases

Returning to the general Kagomé lattice model, the coefficients K_1, K_2, K_3, K'' can be regarded as independent real parameters. Then K_1', K_2', K_3' are uniquely defined by (8.1) and are also real.

The elliptic modulus k is given by (6.4) and (4.8). Using (8.1) to eliminate K_j', and setting

$$\left.\begin{aligned} L_1 &= -K_1+K_2+K_3-K'', \quad & L_2 &= K_1-K_2+K_3-K'', \\ L_3 &= K_1+K_2-K_3-K'', \quad & L_4 &= K_1+K_2+K_3+K'', \end{aligned}\right\} \tag{8.11a}$$

$$\left.\begin{aligned} M_1 &= -K_1+K_2+K_3+K'', \quad & M_2 &= K_1-K_2+K_3+K'', \\ M_3 &= K_1+K_2-K_3+K'', \quad & M_4 &= K_1+K_2+K_3-K'', \end{aligned}\right\} \tag{8.11b}$$

$$l = \cosh L_1, \quad m = \cosh L_2, \quad n = \cosh L_3, \quad p = \cosh L_4, \tag{8.12}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

we find after some lengthy algebra that

$$\Delta = -\tfrac{1}{4}[\cosh M_1 \cosh M_2 \cosh M_3 \cosh M_4]^{-\frac{1}{2}} \times \{\sinh 2K'' + 2\tanh 2K'' \cosh 2K_1 \cosh 2K_2 \cosh 2K_3 + 2\sinh 2K_1 \sinh 2K_2 \sinh 2K_3\}, \quad (8.13)$$

$$\frac{1-k^2}{k^2} = \frac{(-l+m+n+p)\,(l-m+n+p)\,(l+m-n+p)\,(-l-m-n+p)}{16\cosh M_1 \cosh M_2 \cosh M_3 \cosh M_4}. \quad (8.14)$$

[Also, k^{-2} is given by the right hand side of (8.14), but with each cosh in (8.12) replaced by sinh.]

Using the argument of §7, we expect the model to be in an ordered phase if $0 < k^2 < 1$, disordered otherwise. From (8.11)–(8.14) it follows that there are eight domains in (K_1, K_2, K_3, K'') space in which the system is ordered, namely those in which one of l, m, n, p is greater than the sum of the other three, the corresponding L_j being either positive or negative. There is one domain in which the system is disordered, namely when each of l, m, n, p is less than the sum of the other three.

The archetypal ordered phase is

$$p > l+m+n, \quad L_4 > 0. \quad (8.15)$$

In this domain the spins on each sub-lattice are ferromagnetically ordered. If $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are defined by (6.17), then (6.10) and (6.13) are satisfied. Parameters q, x, z_1, z_2, z_3 can then be defined by (7.23). By using (7.9) and (3.5), the free energy function in (8.2) is then given by

$$-\beta f(K_j, K_j', K'') = K_j + K_j' + K'' + \sum_{n=1}^{\infty} \frac{x^{-n}(x^{2n}-q^n)^2\,(x^n + x^{-n} - z_j^n - z_j^{-n})}{n(1-q^{2n})\,(1+x^{2n})}. \quad (8.16)$$

If the conjectures of Barber & Baxter (1973) and Baxter & Kelland (1974) are valid, then the spontaneous magnetization and polarization are given by (7.10) and (7.11).

The other seven ordered phases can all be mapped to (8.15) by reversing some of the lattice spins. For instance, reversing the spin inside each up-triangle is equivalent to negating K_1, K_2, K_3, K'', while leaving K_1', K_2', K_3' unchanged. This negates $L_1, \ldots, M_4$ and leaves k unchanged. Thus it maps the domain (8.15) to $p > l+m+n$, $L_4 < 0$; and vice versa.

Also, reversing the spins between alternate pairs of the horizontal lines in figure 8 is equivalent to negating K_1, K_2, K_1', K_2', while leaving K_3, K_3', K'' unchanged. From (8.11)–(8.14) this leaves unchanged but maps (8.15) to the domain $n > l+m+p$, $L_3 < 0$; and vice versa.

Similarly, one can reverse all the spins between alternate pairs of parallel diagonal lines in figure 8, thereby mapping (8.15) to either $l > m+n+p$, $L_1 < 0$, or to $m > l+n+p$, $L_2 < 0$. These domains can then be mapped to the domains with corresponding $L_j > 0$ by further reversing the spins inside each up-triangle.

The disordered domain is

$$l, m, n, p < \tfrac{1}{2}(l+m+n+p).$$

(Note that these inequalities are certainly satisfied if the interaction coefficients are all small.) In this domain M and P must be zero. The free energy can be obtained from (8.2), by using the definitions (3.5), (7.1)–(7.9) of the function $-\beta f(K_j, K_j', K'')$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Critical behaviour

Since the restrictions (8.1) can in general only be satisfied for a few discrete values of the temperature (if any), we cannot discuss the temperature dependence of the model. Nevertheless, we can define a parameter which plays the same rôle, namely

$$t = (k^2-1)/k^2. \tag{8.17}$$

This is positive for the disordered phase, negative for ordered ones, and vanishes linearly on a path in (K_1, K_2, K_3, K'') space as a critical surface is crossed non-tangentially.

Suppose one starts in the ordered ferromagnetic phase and approaches the critical surface $p = l+m+n$, $L_4 > 0$. Then k^2 tends to one from below. From (6.17), $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \lambda$ and $\mathscr{K}'$ all remain finite and analytic, while $\mathscr{K}$ diverges, being given by

$$\mathscr{K} = \tfrac{1}{2}\ln[-16/t] + \text{vanishing terms}. \tag{8.18}$$

From (7.23), q, x, z_1, z_2, z_3 therefore all tend to one, and the expressions (7.9)–(7.11) for f, M, P become very slowly convergent. It is then appropriate to make a Poisson transform of these series and products (as is explained in Baxter (1972), Barber & Baxter (1973), Baxter & Kelland (1974)) Doing this, the dominant singular contribution to $-\beta f$ is found to be given by

$$(-\beta f)_{\text{sing}} \propto \exp(-2\pi\mathscr{K}/\lambda), \tag{8.19}$$

while M and P are given by

$$M \simeq 2^{\frac{1}{2}}\exp(-\pi\mathscr{K}/8\lambda), \tag{8.20}$$

$$P \simeq (2\mathscr{K}'/\lambda)\exp[-\pi\mathscr{K}(2\mathscr{K}'-\lambda)/4\lambda\mathscr{K}']. \tag{8.21}$$

From (6.17d), $\mathscr{K}' = \frac{1}{2}\pi$ at criticality. From (8.18) it follows that

$$(-\beta f)_{\text{sing}} \propto (-t)^{\pi/\lambda}, \tag{8.22}$$

$$M \propto (-t)^{\pi/16\lambda}, \tag{8.23}$$

$$P \propto (-t)^{(\pi-\lambda)/4\lambda}. \tag{8.24}$$

Thus the critical exponents are the same as those for a corresponding square lattice, namely

$$\alpha = 2-\pi/\lambda, \quad \beta_M = \pi/16\lambda, \quad \beta_e = (\pi-\lambda)/4\lambda, \tag{8.25}$$

λ having the same value at criticality as the parameter μ or $\bar{\mu}$ used in earlier papers. It is defined by (6.17c). At criticality $k^2 = 1$, so λ is then given by

$$\cos\lambda = -\tanh 2K'', \quad 0 < \lambda < \pi. \tag{8.26}$$

For the Ising model case, $K'' = 0$ and $\lambda = \mathscr{K}' = \pi/2$.

This formula applies on the critical surface $p = l+m+n$, $L_4 > 0$, when $\Delta = -1$, so can be replaced by

$$\cos\lambda = \Delta^{-1}\tanh 2K'', \quad 0 < \lambda < \pi. \tag{8.27}$$

From (8.13), $\Delta^{-1}\tanh 2K''$ is unchanged by the various mappings between the eight ordered states described above, so (8.27) and (8.25) are valid on all eight critical surfaces of the Kagomé lattice model. By using (7.12) and (7.13), (8.27) can be written in terms of the Boltzmann weights of a site of type j as

$$\cos\lambda = 2(c_j d_j - a_j b_j)/(a_j^2+b_j^2-c_j^2-d_j^2),$$
$$0 < \lambda < \pi. \tag{8.28}$$

On a critical surface, one of a_j, b_j, c_j, d_j is equal to the sum of the other three.

9. Two-spin correlations

Arguments similar to those of § 5 can be used to establish remarkable equivalences between the two-spin correlations of Z-invariant lattice models.

Consider any two faces l and n of an arbitrary lattice $\mathscr{L}$, as in figure 11. Construct a Z-invariant eight-vertex model on $\mathscr{L}$, with given values of K'' and Δ, and consider the correlation $\langle \sigma_l \sigma_n \rangle$. Suppose that l and n are deep within $\mathscr{L}$, so that boundary conditions are irrelevant.

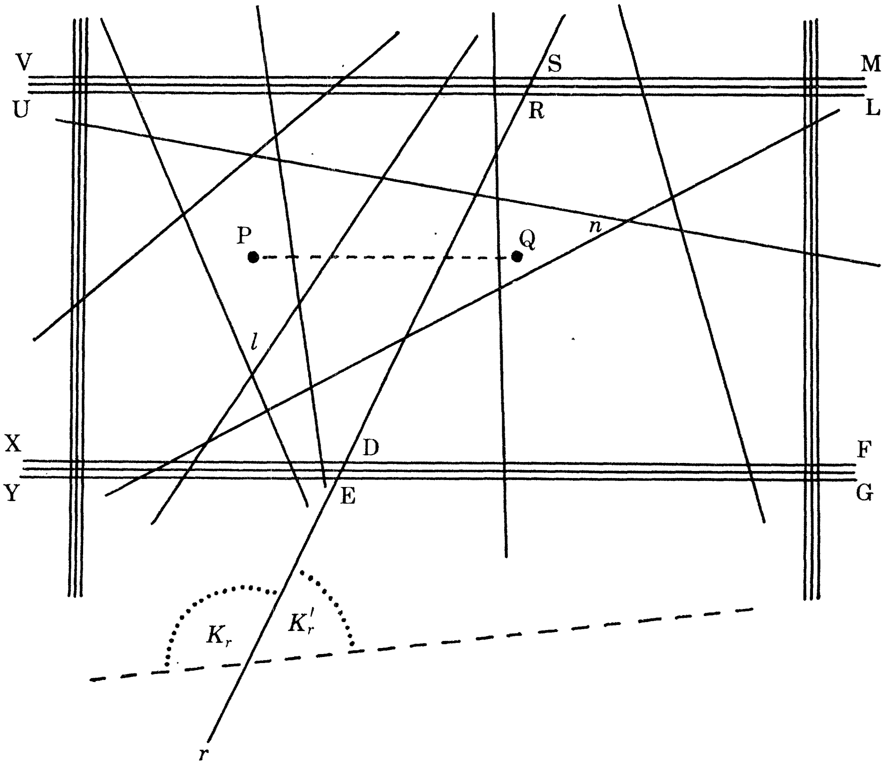

Figure 11. A lattice $\mathscr{L}$ extended by adding a rectangular frame aligned to the line PQ between faces l and m. The broken line is a base line of $\mathscr{L}$.

Let P be a point inside l, Q a point inside m, chosen so that the line PQ does not pass through any lattice site. Orient the lattice so that PQ is horizontal. The line segment PQ intercepts some of the lattice lines. Label these 1, 2, ..., m. Let K_r', K_r be the angle coefficients of the intersection of line r with some base line, ordered as in figure 11. Let α_r, β_r be the corresponding elliptic angle parameters, defined in the ferromagnetic phase by (6.17), (6.5) and (6.6). Thus $\alpha_r + \beta_r = \lambda$, and α_r is the 'line angle parameter' (6.20).

We shall show that $\langle \sigma_l \sigma_n \rangle$ is a function only of m, K'', Δ and $K_1', \ldots, K_m'$, $K_1, \ldots, K_m$, the function being *independent of the structure of* $\mathscr{L}$. In particular, it is unchanged by simultaneously interchanging K_i' with K_j', K_i with K_j.

By using the elliptic parameters of § 6, this implies that

$$\langle \sigma_l \sigma_n \rangle = g_m(k, \lambda; \alpha_1, \ldots, \alpha_m), \tag{9.1}$$

where the function g_m is the same for all Z-invariant models, and is a symmetric function of $\alpha_1, \ldots, \alpha_m$. Adding the same constant to each of $\alpha_1, \ldots, \alpha_m$ re-defines the coefficients on the base line, but from (6.21) leaves the coefficients at the sites of $\mathscr{L}$ unchanged. Thus g_m must be a function only of the differences of $\alpha_1, \ldots, \alpha_m$.

Note that g_1 is the correlation between two adjacent spins, i.e. the polarization. We have already seen that this is a function only of k and λ.

To establish these results, first extend $\mathscr{L}$ by adding $2M$ lines parallel to PQ, M being close together and above $\mathscr{L}$, the others being below $\mathscr{L}$, as in figure 11. Choose the angle coefficients of the intersections of these lines with lattice line r to be K_r', K_r, for $r = 1, \ldots, m$. For example, in figure 11

$$K_{\mathrm{RDF}} = K_{\mathrm{REG}} = K_{\mathrm{DRU}} = K_{\mathrm{DSV}} = K_r',$$
$$K_{\mathrm{RDX}} = K_{\mathrm{REY}} = K_{\mathrm{DRL}} = K_{\mathrm{DSM}} = K_r.$$

This is consistent with Z-invariance.

Further extend $\mathscr{L}$ by adding $2M$ vertical lines, M to the left of $\mathscr{L}$ and M to the right of $\mathscr{L}$. Thus there are $4M$ lines forming a rectangular 'frame' around $\mathscr{L}$. At all intersections of framing lines assign the same coefficient K_0 to the top-left and bottom-right angles, K_0' to the other two angles, choosing K_0, K_0' to satisfy (4.8), with $\Delta_j = \Delta$.

Extend all lattice lines (rotating them slightly if necessary) to cross all the framing lines. Extend the convex boundary outwards to include all these intersections. Assign coefficients to new intersections according to the rules of §4.

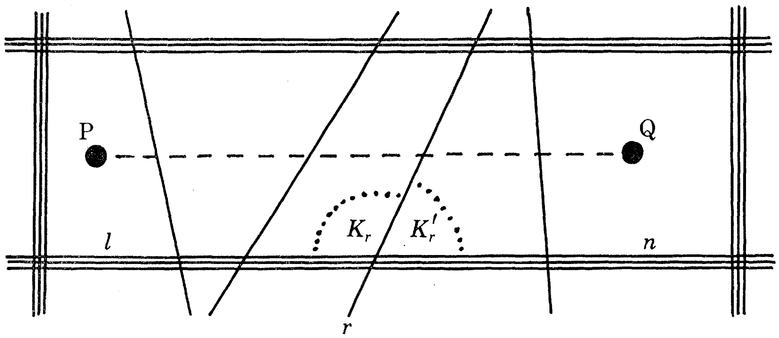

FIGURE 12. The irregular rectangular lattice obtained by shifting the framing lines of figure 11 inwards towards P and Q, and then neglecting sites outside the framing lines.

Suppose this can be done without introducing non-positive Boltzmann weights (for the geometric model of §6 this is certainly so). Then $\langle\sigma_l \sigma_n\rangle$ should be the same for this extended lattice as for the original lattice $\mathscr{L}$, provided l and n were originally deep within $\mathscr{L}$.

The extended lattice is by construction Z-invariant. Shift the framing lines inwards towards PQ until all sites of $\mathscr{L}$ lie outside the framing lines, as indicated in figure 12. This does not change the order of the lines at the boundary, nor does any line cross face l or face n. From §4 it follows that $\langle\sigma_l \sigma_n\rangle$ is unchanged.

However, the picture is now quite different. From figure 12, l and n lie in the same row of a rectangular lattice of $2M$ rows and $2M$ columns. The coefficient of the top-left (top-right) angle is K_r (K_r') for a site on column and between P and Q. It is K_0 (K_0') for a site on a column to the left of P, or to the right of Q. The four-spin coefficient K'' and the parameter Δ are of course the same as in the original lattice $\mathscr{L}$.

If M is large, we expect $\langle\sigma_l \sigma_n\rangle$ to be unchanged by deleting all sites outside this rectangular lattice, so $\langle\sigma_l \sigma_n\rangle$ is the correlation between two spins in the same row of a rectangular lattice model. The model is not regular, since the two-spin coefficients vary from column to column.

Further, in the extended lattice we could have also made parallel shifts of lines 1, ..., m to re-order their intersections with the line PQ. This also leaves the boundary order unchanged, and no line crosses face l or face n, so $\langle\sigma_l\,\sigma_n\rangle$ is unchanged. Hence $\langle\sigma_l\,\sigma_n\rangle$ is independent of the order of columns 1, ..., m in figure 12. This establishes the assertions made above, in particular equation (9.1).

Intra-row correlations in the Kagomé lattice

In specific cases it may be possible to simplify the above argument. For instance, consider the correlation $\langle\sigma_l\,\sigma_n\rangle$, where l and n are the faces of the Kagomé lattice shown in figure 8. By considering a finite, but large, lattice with extended external edges, it becomes apparent that the horizontal lines above (below) l and n can be shifted far upwards (downwards), while leaving $\langle\sigma_l\,\sigma_n\rangle$ unchanged. Thus $\langle\sigma_l\,\sigma_n\rangle$, and any correlation between spins in the same row, is the same as if all horizontal lines in figure 8 were deleted. This leaves a regular square lattice drawn diagonally, with coefficients K_3, K_3', K''. Thus intra-row correlations for the Kagomé lattice are the same as those of this square lattice.

For the six-vertex models this has been established directly by Baxter, Temperley & Ashley (1977, §2.3) using the results of Kelland (1974).

Since the correlation length ξ of the square lattice model diverges when $k^2 = 1$ (Johnson, Krinsky & McCoy 1973), that of the Kagomé lattice model of section 8 must do so in the same way, i.e.

$$\xi \propto (-t)^{-\pi/2\lambda}, \tag{9.2}$$

where at criticality λ is defined by (8.27) or (8.28).

Commuting transfer matrices

Just as the intra-row correlations of the Kagomé lattice model depend only on K_3, K_3', K'' so do the elements of the maximal eigenvector of the transfer matrix, provided, the lattice is infinitely large.

More strongly, impose cylindrical boundary conditions on the lattice, linking the right side to the left, and consider a finite Kagomé lattice model where K_1, K_1', K_2, K_2' can vary from row to row, but K_3, K_3', K'' are constant and the star-triangle relations (8.1) are satisfied for all up-triangles. Then the model is still Z-invariant, and one can establish that interchanging two horizontal lines (together with their associated values of K_1, K_1', K_2, K_2') leaves Z unchanged. It also leaves unchanged all correlations not involving the spins inbetween the two lines.

This implies that the transfer matrices associated with the two lines commute. In particular, they commute when K_3 becomes infinite and the model becomes a square lattice model, as explained in the text following equation (8.3).

Establishing this commutation property was the first step in the original solution of the square-lattice eight-vertex model (Sutherland 1970; Baxter 1972).

Disordered ferromagnetic phase

A particularly important case of the disordered phase, for any lattice $\mathscr{L}$, is when at each site the inequality (6.14) is *not* valid, but (6.25) is. In terms of the vertex weights (3.5) this is the domain

$$a_j+b_j+d_j > c_j > a_j+b_j-d_j, \quad c_j > |a_j-b_j|+d_j \tag{9.3}$$

i.e. $w_1, \ldots, w_4$ in (7.1) are ordered so that $w_1 > w_3 > w_2 > |w_4|$.

If the interaction coefficients are all positive, then this is the only disordered case that can occur. From (7.13), (7.25) and (7.26), in this case

$$-1 < \Delta < 0, \quad k^2 > 1, \tag{9.4}$$

where Δ, k are defined by (4.8) and (6.4).

Interchanging w_2 and w_3 is equivalent to making the duality transformation (3.7). Thus if we first make this transformation and then re-define k, α_j, β_j, λ, $\mathscr{K}'$, $\mathscr{K}$ by (6.4), and (6.8) or (6.17), then the relations (7.23) will still be valid, and the free energy function will be given by (7.9), with $c' = c^* = \frac{1}{2}(a+b+c+d)$.

Let k^*, α_j^*, β_j^*, $\mathscr{K}^{*\prime}$, $\mathscr{K}^*$ be the new elliptic parameters defined by this procedure. The duality transformation (3.7) inverts k, while leaving $k^{\frac{1}{2}}\sinh 2K''$, $k^{\frac{1}{2}}\sinh 2K_j'$, $k^{\frac{1}{2}}\sinh 2K_j$ unchanged. From (6.4) and (6.8) it follows that

$$k^* = \{\Delta^2\cosh^2 2K'' - \sinh^2 2K''\}^{\frac{1}{2}}, \tag{9.5}$$

$$\left.\begin{aligned} \sinh 2K'' &= -\mathrm{i}/\mathrm{sn}\,(\mathrm{i}\lambda^*, k^*), \\ \sinh 2K_j' &= \mathrm{i}/\mathrm{sn}\,(\mathrm{i}\alpha_j^*, k^*), \\ \sinh 2K_j &= \mathrm{i}/\mathrm{sn}\,(\mathrm{i}\beta_j^*, k^*), \end{aligned}\right\} \tag{9.6}$$

where $0 < k^* < 1$ and λ^*, α_j^*, β_j^* are all real, lying in the interval $(0, 2\mathscr{K}^{*\prime})$.

By noting that $k^* = k^{-1}$ and $\mathrm{sn}\,(u, k^{-1}) = k\,\mathrm{sn}\,(k^{-1}u, k)$ (Gradshteyn & Ryzhik 1965, §8.152), and comparing (9.6) with (6.8), it follows that

$$\lambda^* = k\lambda, \quad \alpha_j^* = k\alpha_j, \quad \beta_j^* = k\beta_j, \tag{9.7}$$

where λ, α_j, β_j are defined by (6.17). Thus the elliptic parameters in the ferromagnetic disordered phase are proportional to the analytic continuation of those in the ordered phase. The Z-invariance conditions (i), (ii) and (iii) are therefore unaltered. In particular, at each site j of $\mathscr{L}$ we must have

$$\alpha_j^* + \beta_j^* = \lambda. \tag{9.8}$$

Ising model case: $K'' = 0$

Unfortunately the correlation functions g_m have not yet been evaluated for general values of K'' (apart from g_2, which can in principle be obtained by differentiating the free energy).

They can be obtained when $K'' = 0$, since the model then factors into two independent Ising models. For even m the functions g_m can be obtained by generalizing the Pfaffian method of Montroll, Potts & Ward (1963). In particular, in the high-temperature disordered phase

$$g_2(k; \alpha_1, \alpha_2) = \frac{4\pi}{k^*\mathscr{K}^*}\sum_{n=1}^{\infty}\frac{p^{2n-1}\cosh\left[(n-\frac{1}{2})\,\pi(\alpha_1^*-\alpha_2^*)/\mathscr{K}^*\right]}{(1+p^{2n-1})^2}, \tag{9.9}$$

where

$$p = \exp\left(-\pi\mathscr{K}^{*\prime}/\mathscr{K}^*\right), \tag{9.10}$$

k^*, $\mathscr{K}^*$, $\mathscr{K}^{*\prime}$, are the elliptic parameters defined by (9.5) and (4.8), i.e. (using $K'' = 0$)

$$k^* = \sinh 2K_j \sinh 2K_j', \tag{9.11}$$

and α_1^*, α_2^* are the elliptic line parameters defined by (6.20) (with an asterisk on each α). For $m = 2$ there are two lines between spins σ_l and σ_n, and $|\alpha_1^* - \alpha_2^*|$ is the elliptic parameter of the angle between these lines that includes neither face l nor face n.

When $K'' = 0$ we see from (9.6) that $\lambda^* = \mathcal{K}^{*\prime}$, so the dependence of g_m on λ can be suppressed, and α_1^*, α_2^* must lie in the interval $(0, \mathcal{K}^{*\prime})$.

For the rest of this section we consider elliptic functions of modulus k^*. We omit the asterisks on k, k', $\mathcal{K}$, $\mathcal{K}'$, α_j. The formula (9.9) can be written as

$$g_2(k;\alpha_1,\alpha_2) = -\frac{2\mathcal{K}}{\pi}\left(\frac{k'}{k}\right)^{\frac{1}{2}}\frac{\Theta_1'(\mathrm{i}\alpha_1-\mathrm{i}\alpha_2)}{H(\mathrm{i}\alpha_1-\mathrm{i}\alpha_2)}, \tag{9.12}$$

$$= -\frac{k'^{\frac{1}{2}}\Theta^2(0)}{2\pi}\int_{-\mathcal{K}}^{\mathcal{K}}\frac{\Theta(\mathrm{i}\alpha_1-\mathrm{i}\alpha_2)\,\Theta_1(2s-\mathrm{i}\alpha_1-\mathrm{i}\alpha_2)}{h(s-\mathrm{i}\alpha_1)\,h(s-\mathrm{i}\alpha_2)}\,\mathrm{d}s, \tag{9.13}$$

where H, Θ, H_1, Θ_1 are the Jacobi theta functions (Gradshteyn & Ryzhik 1965, §§ 8.191 and 8.192), and

$$h(u) = H(u)\,\Theta(u). \tag{9.14}$$

For general even m, the integral form (9.13) can be generalized to

$$g_m(k;\alpha_1,\ldots,\alpha_m) = \frac{1}{(m/2)!\,H_1(0)}\left[-\frac{k^{\frac{1}{2}}\Theta^3(0)}{2\pi}\right]^{m/2}\int_{-\mathcal{K}}^{\mathcal{K}}\cdots\int\frac{\prod\limits_{1\leqslant j<l\leqslant m}\Theta(\mathrm{i}\alpha_j-\mathrm{i}\alpha_l)\prod\limits_{1\leqslant j<l\leqslant\frac{1}{2}m}h^2(s_j-s_l)}{\prod\limits_{r=1}^{m}\prod\limits_{j=1}^{\frac{1}{2}m}h(s_j-\mathrm{i}\alpha_r)}$$

$$\times\,\psi[2(s_1+\ldots+s_{\frac{1}{2}m})-\mathrm{i}\alpha_1-\ldots-\mathrm{i}\alpha_m]\,\mathrm{d}s_1\ldots\mathrm{d}s_{\frac{1}{2}m}, \tag{9.15}$$

where
$$\psi(u) = \Theta_1(u) \quad\text{if}\quad \tfrac{1}{2}m \text{ odd},$$
$$= H_1(u) \quad\text{if}\quad \tfrac{1}{2}m \text{ even}. \tag{9.16}$$

This is a rather unwieldy formula, but it does explicitly exhibit the fact that g_m is a symmetric function of $\alpha_1, \ldots, \alpha_m$. A useful recurrence relation is

$$g_m(k;\alpha_1,\ldots,\alpha_{m-2},\alpha_{m-1},\alpha_{m-1}+\mathcal{K}') = g_{m-2}(k;\alpha_1,\ldots,\alpha_{m-2}),\quad g_2(k;\alpha,\alpha+\mathcal{K}')=1. \tag{9.17}$$

Note that *any* two-spin correlation for any Z-invariant lattice Ising model (e.g. the triangular Ising model) must be of the form (9.15), with an appropriate choice of the parameters $\alpha_1, \ldots, \alpha_m$ of the intermediate lines.

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Appendix A. Proof of quadrilateral theorem

First consider a typical triangle ABC as in figure 5. Use the notation of equation (4.1) and define using (4.6),

$$\left.\begin{aligned} a &= e^{2K_1}, \quad b = e^{2K_2}, \quad c = e^{2K_3}, \\ a' &= e^{2K_1'}, \quad b' = e^{2K_2'}, \quad c' = e^{2K_3'}, \\ t &= e^{-2K''}. \end{aligned}\right\} \tag{A 1}$$

(These a, b, c, a', b', c' are *not* the Boltzmann weights used in the text.) Then the star-triangle relations (4.4) are

$$b'c' = (1+tabc)/(bc+ta), \tag{A 2a}$$

$$c'a' = (1+tabc)/(ca+tb), \tag{A 2b}$$

$$a'b' = (1+tabc)/(ab+tc). \tag{A 2c}$$

Solving (A 2*a*) for a gives
$$a = t^{-1}(bcb'c'-1)/(bc-b'c'). \tag{A 3}$$

Alternatively, taking ratios of (A 2*b*) and (A 2*c*) to eliminate a', then solving for a, gives

$$a = t(bc'-b'c)/(bb'-cc'). \tag{A 4}$$

Eliminating a between (A 3) and (A 4) gives

$$\Delta(b,b') = \Delta(c,c'), \tag{A 5}$$

where
$$\Delta(b,b') = \tfrac{1}{2}[t^2(b^2+b'^2)-1-b^2b'^2]/[(1+t^2)\,bb']. \tag{A 6}$$

This is the equation $\Delta_B = \Delta_C$ of equation (4.7) and (4.8) of the text. Note that if it is satisfied, then (A 2*a*) is a corollary of (A 2*b*) and (A 2*c*).

Substituting the expression (A 3) for a into the numerator on the right hand side of (A 2*b*), and the expression (A 4) into the denominator, gives

$$a' = t^{-1}(bb'-cc')\,(b^2c^2-1)/[(bc-b'c')\,(b^2-c^2)]. \tag{A 7}$$

To summarize so far: the a, b, c, a', b', c' are Boltzmann weights associated with the angles of the triangle ABC; a, b, c are associated with the interior angles, while a', b', c' are associated with the complementary exterior angles. If the star-triangle relations (A 2) are satisfied, and b, b', c, c', t are known, then a can be obtained from either (A 3) or (A 4), and a' from (A 7). The b, b', c, c', t must satisfy (A 6).

Now consider the quadrilateral ABCDEF shown in figure 5. Let a, b, c, a', b', c', d, e, f, u, v, w be the weights associated with the indicated angles, e.g. $w = \exp(2K_{\text{EFB}})$.

Suppose the star-triangle relations are satisfied for the triangles AEF, BFD, CDE. Then from (4.6), $K''_A, \ldots, K''_F$ all have the same value K'', so t is a constant weight. For the triangle AEF the relations (A 3), (A 4), (A 7) give

$$a' = t^{-1}(efvw-1)/(ew-fv), \tag{A 8}$$

$$a' = t(ef-vw)/(ev-fw), \tag{A 9}$$

$$a = t^{-1}(ev-fw)\,(e^2w^2-1)/[(ew-fv)\,(e^2-w^2)]. \tag{A 10}$$

For the triangle BFD they give

$$b = t^{-1}(dfuw-1)/(dw-fu), \tag{A 11}$$

$$b = t(df-uw)/(du-fw), \tag{A 12}$$

$$b' = t^{-1}(du-fw)\,(d^2w^2-1)/[(dw-fu)\,(d^2-w^2)]. \tag{A 13}$$

For CDE, the relation (A 4) gives

$$c = t(de-uv)/(ev-du). \tag{A 14}$$

We want to prove that the star-triangle relations are necessarily satisfied for the triangle ABC. In particular, we want to establish the relation (A $2c$), i.e.

$$tc(ab-a'b') = aba'b'-1. \tag{A 15}$$

To do this, substitute the expressions (A 8), (A 10), (A 11), (A 13), (A 14) for a', a, b, b', c into the left hand side, and the expressions (A 9), (A 10), (A 12), (A 13) for a', a, b, b' into the right hand side. Multiplying out all denominator factors, (A 15) will be satisfied if $J = 0$, where J is the expression

$$\begin{aligned} J \equiv {} & (de-uv)\,[(ev-fw)\,(dfuw-1)\,(d^2-w^2)\,(e^2w^2-1) \\ & -(du-fw)\,(efvw-1)\,(e^2-w^2)\,(d^2w^2-1)] \\ & -(ev-du)\,[(ef-vw)\,(df-uw)\,(d^2w^2-1)\,(e^2w^2-1) \\ & -(ew-fv)\,(dw-fu)\,(d^2-w^2)\,(e^2-w^2)]. \end{aligned} \tag{A 16}$$

Note that J does not explicitly depend on t, which is a slight simplification. The choice of whether to use (A 8) or (A 9), (A 11) or (A 12) has been made to ensure this.

The expression J is a polynomial in w of degree six. Setting $w^2 = \pm 1$, we find that J vanishes, so it contains a factor w^4-1. Now it is not too difficult to verify that

$$J \equiv de\,(w^4-1)\,L, \tag{A 17}$$

where

$$\begin{aligned} L = {} & du\,[(e^2+v^2)\,(1+f^2w^2)-(f^2+w^2)\,(1+e^2v^2)] \\ & +ev\,[(f^2+w^2)\,(1+d^2u^2)-(d^2+u^2)\,(1+f^2w^2)] \\ & +fw\,[(d^2+u^2)\,(1+e^2v^2)-(e^2+v^2)\,(1+d^2u^2)]. \end{aligned} \tag{A 18}$$

The d, e, f, u, v, w are not independent, since from the corollary (A 5) of the star-triangle relations, applied to the triangles AEF, BFD, CDE, they must satisfy:

$$\Delta(d,u) = \Delta(e,v) = \Delta(f,w). \tag{A 19}$$

Using the form (A 6) of the function $\Delta(b,b')$, the equations (A 19) are linear in t^2. Eliminating t^2 gives the equation

$$L = 0, \tag{A 20}$$

where L is defined by (A 18). From (A 17) it follows that J does vanish, and hence the relation (A 2c) is satisfied for the triangle ABC in figure 4.

Now interchange d with u, e with w, and f with v in the above working. This leaves (A 19) and (A 20) still satisfied, and from the star-triangle relations for AEF, BFD, CDE, the right hand sides of equations (A 8)–(A 14) become a', a', a, c, c, c', b, respectively. Thus we have also established that

$$tb(ac-a'c') = aca'c'-1, \tag{A 21}$$

which is the relation (A 2b).

Finally, note from (4.9) that the star-triangle relations for AEF, BFD, CDE imply that Δ has the same value at all points of the quadrilateral, and in particular that $\Delta_B = \Delta_C$. From the observation made after equation (A 6), it follows that the relation (A 2a) must also be satisfied.

Thus the star-triangle relations for the triangles AEF, BFD, CDE imply that the star-triangle relations are also satisfied for the triangle ABC, which is the required theorem.

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